Lemma 1. *For every $n \in \mathbb{N}$ there exists $K = n4^n$ orthonormal functions $f_1, \dots, f_K$ such that*

(1) *$f_i(t) = \pm 1$ for all i and all t, and $\int f_i = 0$.*
(2) *For all t except for a set of measure e^{-cn},*

$$\sup_{k \le K} \left| \frac{1}{\sqrt{K}} \sum_{i=1}^{k} f_i(t) \right| \ge c\sqrt{n}.$$

This is quite well known if one does not require $f_i = \pm 1$ or, alternatively, if one allows coefficients i.e. writes $\sum c_i f_i$ for some c_i such that $\sum c_i^2 = 1$. In this formulation I don't have a reference at hand.

Proof. On the set $\mathcal{H} := \{(p, q) : 0 \le p \le n,\ 0 \le q \le 2^p - 1\}$ we define the lexicographical order $\le$ (later on we will need a different order on $\mathcal{H}$). We denote by $k(p, q)$ the position of (p, q) in the order $\le$, in a formula

$$k(p, q) = 1 + q + \sum_{i=0}^{p-1} 2^i$$

(ranging from 1 to $2^{n+1} - 1$). The inverse functions will be denoted by $p(k)$ and $q(k)$ so that $k(p(i), q(i)) = i$. We now define inductively blocks of functions g which will be our f-s after a reordering in the style of Olevskiĭ. More specifically, we construct the following objects:

(1) Two functions $\alpha_k, \beta_k : [0, 1] \to [0, 1]$ which are constant on intervals of the form $[m2^{-4nk}, (m+1)2^{-4nk})$. For every t, $\alpha_k(t) < \beta_k(t)$ and $[\alpha_k(t), \beta_k(t)]$ is a maximal interval of constancy for the first k Haar functions. Note that the first k Haar functions have $k+1$ (maximal) intervals of constancy, $2q(k)+2$ of them of length $2^{-p(k)-1}$ and the rest of length $2^{-p(k)}$. A second property we will preserve is

$$[\alpha_k(t), \beta_k(t)] \subset [\alpha_j(t), \beta_j(t)] \qquad \forall t\, \forall j < k. \tag{1}$$

Finally, α and β will have the property that the measure of the inverse images is just the measure of the interval, namely

$$\mathbf{m}\{t : \alpha(t) = a\} = \mathbf{m}\{t : \beta(t) = b\} = b - a \tag{2}$$

for any interval of constancy $[a, b]$.

(2) Functions $g_{i,j}$ for $i \le k$ and $j \in \{1, \dots, 4^n 2^{-p(i)}\}$ (at the k'th step we define only $g_{k,j}$). The functions are constant on intervals of the form $[m2^{-4kn}, (m+1)2^{-4kn})$, are all orthogonal and $g_{i,j}(t) = \pm 1$. The connection between g, α and β is in (4) below.

Start with $\alpha_0 \equiv 0$ and $\beta_0 \equiv 1$ (as we must). The process is as follows. Assume we are in the k'th step, $k \ge 1$, and denote $p = p(k)$, $q = q(k)$. Let I be the interval $[q2^{-p}, (q+1)2^{-p})$. Write $2^p \mathbf{1}_I$ as a polynomial in Walsh functions. Since (denoting by r_i the i'th Radamacher function)

$$2^p \mathbf{1}_I = \prod_{i=1}^{p} (1 + r_i) \text{ or } (1 - r_i) \text{ depending on the } i\text{'th digit of } q$$

we see that

$$2^p \mathbf{1}_I = \sum_{i=0}^{2^p - 1} \pm W_i$$

where W_i are the Walsh functions. Now write $l = n - p$ and examine the product

$$B := \prod_{i=1}^{l} (r_{n+3i-2} + r_{n+3i-1} + r_{n+3i} - r_{n+3i-2} r_{n+3i-1} r_{n+3i}).$$

A simple check shows that each term in the product is ± 2 for all t and therefore $B(t) = \pm 2^l$. Denote $H = 2^p B \mathbf{1}_I$. This function is equi-distributed like a Haar function, hence the notation H. The expansion of H into Walsh functions has $4^l 2^p = 4^n 2^{-p}$ terms which are all ± 1, so write

$$H = \sum_{i=1}^{4^n 2^{-p}} h_i$$

where each $h_i = \pm$some Walsh function, so they are orthonormal and $h_i(t) = \pm 1$.

With H and h_i defined we can describe the induction step. Define g_i which are, on every interval of length $2^{-4n(k-1)}$, a compressed version of h_i on the interval $[\alpha, \beta]$. In a formula,

$$g_{k,i}(t) = h_i \left(\alpha_{k-1}(t) + \left\langle t 2^{4n(k-1)} \right\rangle (\beta_{k-1}(t) - \alpha_{k-1}(t)) \right) \quad i = 1, \ldots, 4^n 2^{-p}$$

where $\langle x \rangle := x - \lfloor x \rfloor$. It is easy to verify all requirements from the $g_{k,i}$: Clearly $g_{k,i}(t) = \pm 1$. Since h_i are constant on every interval of the form $[m 2^{-4n}, (m+1) 2^{-4n})$, and since $\alpha_{k-1}(t) \in 2^{-n-1}\mathbb{N}$ and $\beta_{k-1}(t) - \alpha_{k-1}(t) = 2^{-r}$ for some r we get that $g_{k,i}$ are constant on every interval of the form $[m 2^{-4nk}, (m+1) 2^{-4nk})$, as required. As for orthogonality, $\int g_{k,i} g_{k,i'} = 0$ for all $i \neq i'$ since $g_{i,k}(t) = h_i(Tt)$ for some *measure-preserving* T — here is where we use (2) which ensures that, if we map the intervals of length 2^{-4nk} inside $\{t : \alpha_{k-1}(t) = a\}$ into the interval $[a, b]$ in some way compatible with the definition of $g_{k,i}$ we get a measure-preserving transformation. Here is one possible explicit formula for T:

$$\begin{aligned} T(t) = \alpha_{k-1}(t) + 2^{-4n} \left\lfloor \left\langle t 2^{4n(k-1)} \right\rangle 2^{4n} \right\rfloor (\beta_{k-1}(t) - \alpha_{k-1}(t)) + \\ + \; 2^{-4n} \mathbf{m}\{ s < 2^{-4n(k-1)} \left\lfloor t 2^{4n(k-1)} \right\rfloor : \alpha_{k-1}(s) = \alpha_{k-1}(t) \} + \\ + 2^{-4nk} \left\langle t 2^{4nk} \right\rangle . \end{aligned} \tag{3}$$

The reader would probably find it easier to prove that some measure-preserving T exists with $g = h \circ T$ himself than to verify that the T in (3) is such.

The fact that $\int g_{k,i} g_{k',i'} = 0$ for $k' < k$ follows from the fact that $\int_a^b h_i = 0$ for every interval of constancy $[a, b]$. Therefore we get

$$\int_{m 2^{-6n(k-1)}}^{(m+1) 2^{-6n(k-1)}} g_{k,i} = 0 \quad \forall i \leq 4^l 2^p, m < 2^{4n(k-1)}$$

and since $g_{k',i'}$ is constant on each such interval, we get the full orthogonality.

Finally we need to define α_k and β_k. The requirement $[\alpha_k, \beta_k] \subset [\alpha_j, \beta_j]$ (1) implies that we need only modify them for t-s for which $[\alpha_{k-1}(t), \beta_{k-1}(t)] = I$ — note that I is exactly the interval that gets split into two intervals of constancy when you add the k'th Haar function. We define

$$\alpha_k(t) = \begin{cases} \alpha_{k-1}(t) & \text{when } \sum_{i=1}^{4^n 2^{-p}} g_{k,i}(t) = 2^n \\ \alpha_{k-1}(t) + 2^{-p-1} & \text{when } \sum_{i=1}^{4^n 2^{-p}} g_{k,i}(t) = -2^n \end{cases}$$

and then $\beta_k(t) = \alpha_k(t) + 2^{-p-1}$. As already explained, $g_{k,i}(t) = h_i(Tt)$ for a measure-preserving transformation T. Therefore $\sum g_{k,i}(t) = \sum h_i(Tt)$ and $\sum h_i \neq 0$ exactly on I and there $\mathbf{m}\{t : \sum h_i(t) = \pm 2^n\} = 2^{-p-1}$ (2^n is a product of 2^l from the definition of B and a 2^p from the definition of H). Thus (2) holds for α_k and β_k and the induction is complete.

Before going on let us note the equality that we worked so hard to get.

$$\sum_j g_{k,j} \quad = \quad \begin{cases} 2^n & \text{when } \alpha_k(t) \in [q2^{-p}, (q+\frac{1}{2})2^{-p}) \\ -2^n & \text{when } \alpha_k(t) \in [(q+\frac{1}{2})2^{-p}, (q+1)2^{-p}) \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The reason we wrote the conditions on $\alpha_k(t)$ in (4) so strangely is that in this notation one can replace α_k with $\alpha_{k'}$ for any $k' > k$ — indeed from (1) we have that if $\alpha_k(t) = q2^{-p}$ then $\alpha_{k'}(t) \in [\alpha_k(t), \beta_k(t)) = [q2^{-p}, (q+\frac{1}{2})2^{-p})$ for all $k' \geq k$.

Examine now the situation at last step ($k = 2^{n+1} - 1$). Again we use the the fact that (2) implies that the function α_k defines a measure-preserving transformation T which collects all t with $\alpha_k(t) = a$ into $[a, b]$. Formally, we define

$$T(t) = \alpha_k(t) + \mathbf{m}\{s : s < t,\ \alpha_k(s) = \alpha_k(t)\} \tag{5}$$

and get that $T : [0,1] \to [0,1]$ is one-to-one, onto and measure-preserving (this is more-or-less the same T we had above — before the definition was complicated because we didn't have α_k defined yet, only α_{k-1}). Further,

$$\sum_{j=1}^{4^n 2^{-p(i)}} g_{i,j}(t) = 2^n \eta_i(Tt)$$

where η_i is the i'th Haar function normalized to have $\|\eta_i\|_\infty = 1$. Collect all the functions $g_{i,j}$ (there are exactly $n4^n$ of them) and arrange the blocks $g_{i,j}$ according to the Olevskiĭ order $\preceq$, see [O75, §III.2] and call the result f_i. We get

$$\sup_{k \leq K} \left| \frac{1}{\sqrt{K}} \sum_{i=1}^{k} f_i(t) \right| \geq \sup_{j \leq 2^{n+1}-1} \left| \frac{1}{\sqrt{n}} \sum_{j' \preceq j} h_{j'}(Tt) \right|$$

and as is well known, this last sum is $\geq cn$ except for a set of t-s of measure $\leq e^{-cn}$. This finishes the lemma. □

Theorem. *There exists a sequence of functions φ_i which is orthonormal, stationary and pairwise independent but does not satisfy Carleson's theorem.*

Proof. Define inductively $N_1 = 4$ and $N_i = 2^{N_{i-1}} 4^{2^{N_{i-1}}}$. Examine the following Markov chain. The state space V is

$$\{i, j : i \in \mathbb{N}, 1 \leq j \leq N_i\}.$$

As for the transition probabilities, let

$$q(n) = \frac{1/n^2 N_n}{\sum_{i=1}^{\infty} 1/i^2 N_i}$$

and define

$$p((i_1,j_1),(i_2,j_2)) = \begin{cases} 1 & i_1 = i_2,\ j_2 = j_1 + 1 \\ q(i_2) & j_1 = N_{i_1},\ j_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that

$$\pi(i,j) = \frac{1/i^2 N_i}{\sum_{k=1}^{\infty} 1/k^2}$$

is a stationary probability measure for our process. The reverse process has transition probabilities

$$p^*((i_1,j_1),(i_2,j_2)) = \begin{cases} 1 & i_1 = i_2,\ j_2 = j_1 - 1 \\ q(i_2) & j_1 = 1,\ j_2 = N_{i_2} \\ 0 & \text{otherwise.} \end{cases}$$

and has the same stationary measure. This allows to construct a stationary process M on $\mathbb{Z}$ (formally $M : \Omega \to V^{\mathbb{Z}}$ where Ω is the probability space) by taking $M(0)$ to be distributed as π and then $M(n)$ defined by

$$\begin{aligned} \mathbb{P}(M(n) \,|\, M(n-1)) &= p(M(n-1), M(n)) && n > 0 \\ \mathbb{P}(M(n) \,|\, M(n+1)) &= p^*(M(n+1), M(n)) && n < 0. \end{aligned}$$

Clearly M is a stationary process. Let Ω be a standard probability space realizing M.

We now move to define the φ_i. Clearly one may construct the φ_i on any standard probability space, and we will do it on $\Omega \times \mathbb{T}^{\mathbb{Z}}$. Use lemma 1 for $n = 2^{N_{i-1}}$ and get N_i functions $f_{i,1}, \dots, f_{i,N_i}$. Now define on Ω the process B that counts how many "blocks" were seen, namly

$$B(n) = \begin{cases} \text{number of times } t \in (0,n] \text{ such that } M_2(t) = 1 & n \geq 0 \\ -\text{number of times } t \in (-n,0] \text{ such that } M_2(t) = 1 & n < 0. \end{cases}$$

(here M_2 is the second coordinate, "the j"). Finally write

$$\varphi_n(\omega; \dots, t_{-1}, t_0, t_1, \dots) = f_{M_1(\omega;n), M_2(\omega;n)}(t_{B(\omega;n)}).$$

It remains to prove the promised properties of the φ_n.

The first thing to note is that $\varphi_n = \pm 1$. The next thing is that $\int \varphi_n = 0$. This is clear because

$$\int \varphi_n = \mathbb{E}\left(\int \varphi_n \,\middle|\, \omega \right) = \mathbb{E}\left(\int f_{M_1(\omega;n), M_2(\omega;n)}(t_{B(\omega;n)}) \,\middle|\, \omega \right)$$

and this is 0 because $\int f_{i,j}(t_k) = 0$ for any i, j and k. Similarly we get that the functions are orthogonal:

$$\int \varphi_n \varphi_{n'} = \mathbb{E}\left(\int f_{M_1(\omega;n), M_2(\omega;n)}(t_{B(\omega;n)}) f_{M_1(\omega;n'), M_2(\omega;n')}(t_{B(\omega;n')}) \,\middle|\, \omega \right). \quad (6)$$

For ω such that $B(\omega;n)\neq B(\omega;n')$ this integral is zero because

$$\int f_{i,j}(t_k)f_{i',j'}(t_{k'}) = \left(\int f_{i,j}(t_k)\right)\left(\int f_{i',j'}(t_{k'})\right) = 0 \quad \forall k\neq k'.$$

For ω such that $B(\omega;n)=B(\omega;n')$ the intgeral in (6) is zero because $\int f_{i,j}f_{i,j'}=0$ whenever $j\neq j'$. Hence we get that the integral in (6) is zero for all ω, and therefore the φ_n are orthogonal. For ± 1 functions this automatically implies that they are pairwise independent.

Stationarity more-or-less follows immediately from the stationarity of M. Indeed, let $S:\Omega\to\Omega$ be the "right shift" i.e. the measure-preserving transformation such that $M(S(\omega);n)=M(\omega;n-1)$. Then we can define on $\Omega\times\mathbb{T}^{\mathbb{Z}}$

$$T(\omega;\{t_k\}) = \begin{cases} (S(\omega);\{t_k\}) & M_2(\omega;0)>1 \\ (S(\omega);\{t_{k-1}\}) & M_2(\omega;0)=1 \end{cases}$$

and it is straightforward to verify that T is measure-preserving and

$$\varphi_n(T(\omega;\{t_k\})) = \varphi_{n-1}(\omega;\{t_k\}) \quad \forall n.$$

Finally we need to demonstrate an L^2 function with almost-everywhere diverging expansion. Examine therefore

$$\sum_{i=1}^{\infty}\frac{1}{N_{i-1}^2\sqrt{N_i}}\sum_{n=N_i}^{N_{i-1}^2N_i}\varphi_n$$

which is in L^2. Examine first ω. Until time $N_{i-1}^2N_i$ there are $\leq N_{i-1}^2N_i$ new blocks, and each such block has probability $\leq C4^{-2^{N_i}}$ to be larger than N_i. Therefore with probability $\geq 1-C4^{-2^{N_i/2}}$ no such block exist (these numbers are huge, and we really don't care — let's just write probability $\geq 1-C2^{-i}$). Next examine the probability that all blocks are small. If all blocks up to $(N_{i-1}^2-1)N_i$ have size $<N_i$ then their sizes are $\leq N_{i-1}$ and we must have had at least $\frac{1}{2}N_{i-1}N_i$ new blocks. At each such event the probability to have a block of size N_i is $\geq c/i^2N_i$ and so we get

$$\mathbb{P}(\text{no block of size } N_i \text{ in the first } \tfrac{1}{2}N_{i-1}N_i \text{ blocks}) \leq$$
$$\leq \left(1-\frac{c}{i^2N_i}\right)^{\frac{1}{2}N_{i-1}N_i} \leq e^{-cN_{i-1}/i^2} \leq C2^{-i}.$$

In total we see that with probability $\geq 1-C2^{-i}$ there exists at least one block of size exactly N_i until time $N_{i-1}^2N_i$.

This finishes the theorem, since, conditioning on the block being number B and in position $[q,q+N_i]$,

$$\sup_{n\leq N_{i-1}^2N_i}\left|\frac{1}{\sqrt{N_i}}\sum_{m=1}^{n}\varphi_m(\omega;\{t_k\})\right| \geq \frac{1}{2}\sup_{n\leq N_i}\left|\frac{1}{\sqrt{N_i}}\sum_{m=1}^{n}f_{i,m}(t_B)\right| \geq c\sqrt{2^{N_{i-1}}}$$

for all t except on a set of t-s of measure $e^{-c2^{N_{i-1}}}$. And we are done. □

REFERENCES

[O75] A. M. Olevskiĭ, *Fourier series with respect to general orthogonal systems*, Translated from the Russian by B. P. Marshall and H. J. Christoffers. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 86. Springer-Verlag, New York-Heidelberg, 1975.