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RANDOM HOMEOMORPHISMS AND FOURIER EXPANSIONS

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Abstract

We prove that a random change of variable in general improves convergence properties of the Fourier expansions, and we give a precise quantitative estimate of the phenomenon.

0 Introduction

0.1. It is well known that the best possible estimate of the Fourier partial sums

$$S_n(f;x) = \sum_{|k|\le n} \widehat{f}(k)e^{2\pi ikx}\,,$$

which is true for any function f continuous on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is

$$\|S_n(f)\|_{C(\mathbb{T})} = o(\log n)\,. \tag{1}$$

This result is closely connected with another classical result, which gives an unimprovable condition for uniform convergence of the Fourier series in terms of the modulus continuity:

$$\omega_f(\delta) = o\left(\log \tfrac{1}{\delta}\right)^{-1}\,. \tag{2}$$

0.2. A suitable change of variable allows one to remove the divergence phenomenon. According to the Pal-Bohr Theorem (see [Ba]), for every real function $f \in C(\mathbb{T})$ one can find a homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ such that the superposition $f \circ \varphi$ belongs to $U(\mathbb{T})$ (the space of all functions with uniformly convergent Fourier series). A stronger result of Kahane and Katznelson [KKat] shows that it can be done simultaneously for a given compact family of f's. These results were obtained by different methods; in each case the homeomorphism φ was defined by a special construction using concrete data about f or about its modulus of continuity.

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Detailed surveys of papers on the influence of individual changes of variable on the fundamental properties of Fourier expansions can be found in [K1], [O1,2].

0.3. In this paper we are interested in the role of a *random* homeomorphism.

We mention that an excellent survey on random Fourier series, starting with classical results of Kolmogorov, Paley and Zygmund and ending with modern papers, can be found in [K2]. Most of these results deal with Fourier series with independent random coefficients. We also mention the important papers, [G] and [Bo], on random rearrangement of orthogonal series and a paper, [Ka], about random corrections of functions.

0.4. Our paper is devoted to a different aspect of random Fourier analysis. For a given $f \in C(\mathbb{T})$ we consider a random change of variable $\varphi : \mathbb{T} \to \mathbb{T}$, independent of f, and we are interested in the "typical" behaviour of the Fourier expansion for the superposition $f \circ \varphi$. What is the best possible estimate

$$\left\|S_n(f \circ \varphi)\right\| = o(\omega(n)),$$

which holds for any f and the "majority" of φ?

Our main idea is as follows: the divergence phenomenon of Fourier series is caused by resonance between the given function f and the Dirichlet kernel D_n. A random perturbation should presumably destroy this "unpleasant" resonance, so one may expect that in "most cases" the superposition $f \circ \varphi$ enjoys a better estimate than (1). The main results confirm this conjecture in a precise, quantitative form.

0.5. The first obstacle on the way is how to understand a "typical" change of variable, or, stated differently, how to introduce a measure in the space Φ of all homeomorphisms φ? A "most natural" measure probably does not exist. In particular, the group Φ being non-compact has no invariant probability measure. We use a stochastic structure on Φ suggested in a more general form by Dubbins and Freedman in [DF] and studied by Graf, Mauldin and Williamson in [GrMWi]. This construction is based on the following assumption: for every couple of integers p, q ($0 \le p < 2^q$, $q = 0, 1, 2, \dots$) the value $\varphi\big(\frac{p+(1/2)}{2^q}\big)$ depends *only* on the values $\varphi\big(\frac{p}{2^q}\big)$ and $\varphi\big(\frac{p+1}{2^q}\big)$ and is distributed uniformly between these numbers. The probability measure $\mathbb{P}$ defined this way has the nice property of "selfsimilarity" on dyadical intervals.

In spite of the fact that the trajectories φ are singular almost surely (see [DF]), on average they preserve the Lebesgue measure m on the circle, i.e. $\mathbb{E}(m\varphi(E)) = mE$; in particular

$$mE = 0 \Leftrightarrow m\varphi(E) = 0 \quad \text{a.s.} \quad \text{(see [GrMWi])}.$$

Another nice property is the smoothness: the homeomorphisms φ and φ^{-1} both almost surely belong to the Hölderian class $H^\alpha(\mathbb{T})$ (with some absolute value $\alpha > 0$). In §1 we prove the last statement, as well as some other necessary properties of the distribution $\mathbb{P}$.

0.6. Now we describe the main results. For a given $f \in C(\mathbb{T})$, $\|f\| \le 1$, $n \in \mathbb{N}$ and $t \in \mathbb{T}$, we study the random variable

$$\mathfrak{S} := S_n(f \circ \varphi; t) = \int_{\mathbb{T}} f\big(\varphi(x)\big) D_n(t-x)\, dx\,.$$

We prove (§2) the following estimate for its distribution, which is uniform with respect to all parameters above,

$$\mathbb{P}(\mathfrak{S} > M) < e^{-e^{cM}} \quad (M > M_0) \tag{3}$$

($c > 0$ and M_0 are absolute constants).

Our approach to the proof starts with an analogy to the classical result on the distribution of the sum

$$\mathfrak{T} = \sum_{k \ge 1} \pm \tfrac{1}{k}\,,$$

with equidistributed independent signs. However, the variables $\varphi(\alpha)$ have much weaker independence properties and this makes our analysis much harder than the classical case.

Inequality (3) implies (§3)

Theorem 3. *For any $f \in C(\mathbb{T})$, the Fourier partial sums satisfies the estimate,*

$$\big\|S_n(f \circ \varphi)\big\| = o(\log\log n) \quad \text{a.s.} \tag{4}$$

In §5 we investigate the asymptotic behaviour of $U(\mathbb{T})$ norms for high-frequency oscillations $e^{i\nu\varphi}$ with a random phase φ, which enables us to prove the sharpness of the previous result:

Theorem 5. *For every given sequence $s(n) = o(\log\log n)$ there exists an $f \in C(\mathbb{T})$ such that*

$$\limsup \tfrac{1}{s(n)} \big\|S_n(f \circ \varphi)\big\| = \infty \quad \text{a.s.}$$

Comparing (1) and (3) one can see to which extent a random perturbation improves convergence properties of the Fourier expansion. We also prove a different version of the same phenomenon: the condition

$$\omega_f(\delta) = o\left(\log\log\tfrac{1}{\delta}\right)^{-1}$$

(compare with (2)) implies $f\circ\varphi \in U(\mathbb{T})$ a.s., and the result is sharp (Theorems 4, 6).

A non-stochastic corollary of Theorem 3: *For any $f \in C(\mathbb{T})$ one can achieve* (4) *by a bi-Hölderian change of variable.* In contrast to this, all known constructions of improving homeomorphisms mentioned in 0.2 are extremely non-smooth.

0.7. Another possible interpretation of a "typical" change of variable comes from Baire categories. However, this approach does not give any positive effect. It turns out that the smoothness condition which guarantees that $f\circ\varphi \in U$ for a residual set of homeomorphisms, is even stronger than (2), and is actually identical to the condition providing that $f\circ\varphi \in U$ for *every* φ.

Denoting for a given modulus of continuity $\omega(\delta)$

$$H^\omega = \left\{f : \omega_f(\delta) = O(\omega(\delta))\right\},$$

in §5 we prove that the condition

$$\sum \tfrac{1}{k}\omega\left(\tfrac{1}{k}\right) < \infty$$

is necessary and sufficient for the implication

$$f \in H^\omega \Rightarrow f\cdot\varphi \in U \quad \text{for a residual set of } \varphi\ .$$

This result is intimately connected with the result of [BW].

0.8. We conclude by noting that it might be interesting to find "random versions" of other problems in Fourier analysis on homeomorphisms of the circle, such as those discussed in [O1,2].

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1 Analysis of Random Homeomorphisms

1.1. In this subsection, we basically follow [GrMWi]. We start with a more rigorous definition of the measure $\mathbb{P}$. Let $X_{n,k}$, $n \geq 1$, $0 < k < 2^n$, k odd, be a sequence of independent, uniform random variables on [0,1] (we denote by Ω a probability space realizing all of them). Define $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(1/2) = X_{1,1}$. Next, define $\varphi(1/4) = X_{2,1}\cdot\varphi(1/2)$ and

$\varphi(3/4) = \varphi(1/2) + X_{2,3} \cdot (1 - \varphi(1/2))$. We continue this process, at each step defining $\varphi(k2^{-n}) = \varphi((k-1)2^{-n}) + X_{n,k} \cdot (\varphi((k+1)2^{-n}) - \varphi((k-1)2^{-n}))$ until we have φ defined on each dyadic rational on the interval $[0,1]$. With a probability 1, φ can be extended to a continuous strictly increasing function [0,1] ([DF, Theorem 4.1]). Thus we have defined a measurable map $X : \Omega \to \Phi$, where Φ is the space of all increasing homeomorphisms of [0,1] with the topology induced by the maximum norm and the map is measurable with respect to the borel sets on Φ. The measure $\mathbb{P}$ can be regarded as being defined on Ω or directly on Φ.

REMARK. The measure thus defined is not translation invariant. In particular, 0 is a fixed point. One can make it translation invariant by composing it with a random rotation from the left, the right or even from both sides (making it, in the last case, invariant from both sides). It is easy to check that all the theorems proven in §1-§4 are valid also for these invariant measures.

The following basic properties of $\mathbb{P}$ are rather intuitive.

LEMMA 1.1. *Let I be a dyadic interval, i.e. $I = [d2^{-l}, (d+1)2^{-l}]$. If we fix the value of φ on its boundaries, the interior behaves like a small copy of $\mathbb{P}$. To be more precise, for $\mathbb{P}$-almost any value of $\varphi(\partial I)$ (this "value" is of course two real numbers, $\varphi(d2^{-l})$ and $\varphi((d+1)2^{-l})$),*

$$\big(\varphi \mid \varphi(\partial I)\big)|_I \sim (\varphi \circ L_I) \cdot |\varphi I| + \varphi(d2^{-l}),$$

where L_I is a linear increasing mapping of I onto $[0,1]$, and $|\varphi I|$ is the length of the interval $\varphi I = \varphi((d+1)2^{-l}) - \varphi(d2^{-l})$. The symbol $\sim$ stands for "has the same distribution as".

This property is called "scaling invariance" and $\mathbb{P}$ is said to be scaling invariant on I, see [GrMWi, Theorem 4.6].

LEMMA 1.2. *$\mathbb{P}$ is time-reversal invariant, i.e. $1 - \varphi(1-x) \sim \varphi(x)$.*

See [GrMWi, Theorem 4.3].

The averages of φ and φ^{-1} are also of some interest to us. The following two equalities are proved in [GrMWi]: $\forall x \in [0,1]$

i) $$\mathbb{E}\varphi(x) = x$$

ii) $$\mathbb{E}\varphi^{-1}(x) =: p(x) = \tfrac{1}{2} + \tfrac{1}{\pi}\arcsin(2x-1)\,.$$

As a consequence, by a standard limit process we get

LEMMA 1.3. *Let us extend the σ-algebra on Ω by adding all $\mathbb{P}$-0 measure sets. Then,*

(i) *Let* $E \subset [0,1]$ *be a Lebesgue measurable set. Then* $\varphi(E)$ *and* $\varphi^{-1}(E)$ *are Lebesgue measurable* $\mathbb{P}$*-a.s.;*

(ii) $m(\varphi(E))$ *and* $m(\varphi^{-1}(E))$ *are random variables on the probability space* Ω*, that is measureable maps* $\Omega \to \mathbb{R}$*, and*

$$\mathbb{E}\big(m\varphi(E)\big) = mE$$
$$\mathbb{E}\big(m\varphi^{-1}(E)\big) = \int_E p'\,dx\,,$$

where m *is the Lebesgue measure.*

(iii) *If* $f \in L^\infty[0,1]$ *then* $f \circ \varphi \in L^\infty$ *a.s. and* $\int_0^1 g \cdot (f\circ\varphi)\,dx$ *is a random variable for any* $g \in L^1([0,1])$.

1.2. We consider the following operator in $L^1[0,1]$,

$$(\Theta f)(x) = \int_x^1 \tfrac{1}{y} f(y)dy\,,$$

or, after a change of variable,

$$(\Theta f)(x) = \int_x^1 \tfrac{1}{y} f\left(\tfrac{x}{y}\right)dy\,.$$

Fubini's theorem gives $\int_0^1 \Theta f\,dx = \int_0^1 f\,dx$ and it follows that $\|\Theta f\|_1 \le \|f\|_1$ (equality takes place iff f has constant sign).

Now we state some fundamental properties of the distribution

$$P_\alpha(x) := \mathbb{P}\big(\varphi(\alpha) < x\big)\,,$$

and the corresponding density function $\rho_\alpha := \frac{d}{dx}P_\alpha(x)$.

Theorem 1. (i) *For every* $\alpha \in (0,1)$*, the function* $P_\alpha(x)$ *is continuous on* $\mathbb{R}$ *and* $\in C^1(0,1)$.

(ii) $\rho_\alpha = \Theta\rho_{2a}$ $\quad(0 < \alpha < 1/2)$.

(iii) $\rho_\alpha(x) = \rho_{1-\alpha}(1-x)$ $\quad(1/2 < \alpha < 1)$.

Proof. First note that (iii) follows from Lemma 1.2. Further, if for some $\alpha < 1/2$, $P_{2\alpha} \in C^1(0,1)$ then $P_\alpha \in C^1(0,1)$ and (ii) holds. Indeed, we have

$$\mathbb{P}\big(\varphi(\alpha) < x\big) = \mathbb{E}\left(\mathbb{P}(\varphi(\alpha) < x) \mid \varphi\left(\tfrac{1}{2}\right)\right)\,.$$

The scaling invariance of $\mathbb{P}$ on the interval $[0,1/2]$ implies that for $\mathbb{P}$-almost any value of $\varphi(1/2)$, $(\varphi(\alpha) \mid \varphi(1/2) = y) \sim y \cdot \varphi(2\alpha)$,

$$P_\alpha(x) = \int_0^1 \mathbb{P}\big(y\cdot\varphi(2\alpha) < x\big)dy = \int_0^1 P_{2\alpha}\left(\tfrac{x}{y}\right)dy = x + \int_x^1 P_{2\alpha}\left(\tfrac{x}{y}\right)dy\,,$$

and differentiation with respect to the parameter x (which can be justified easily) gives (ii) and $P_\alpha \in C^1(0,1)$. So for α of the form $d2^{-l}$, the obvious induction over l together with (ii) and (iii) gives (i) and (ii). Moreover, using the definition of Θ we see that $P_\alpha \in C^2(0,1)$ and we have a uniform (over all dyadical α) estimate, $\min_{\delta<x<1-\delta} \rho_\alpha(x)$, $|\rho'_\alpha(x)| < C(\delta)$. Now we fix $\alpha \in (0,1/2)$ and take dyadical $\alpha_n \to 2\alpha$. Obviously, $P_{\alpha_n} \to P_{2\alpha}$ on [0,1]. Standard compactness arguments imply from the estimate above that $P_{2\alpha} \in C^1$, so we finish the proof.

Certainly, $P_\alpha \in C^\infty(0,1)$ for every $\alpha \in (0,1)$ but we do not need it.

We mention one more property which follows from the previous ones:

(iv) *ρ_α is strictly decreasing if $\alpha < 1/2$ and increasing if $\alpha > 1/2$.*

1.3. Now we prove that a random homeomorphism φ has almost surely a Hölder smoothness ($\in H^\gamma := \{f : \omega_f(\delta) = O(\delta^\gamma)\}$) with some fixed order $\gamma > 0$. We need this result in a slightly more precise form. (Throughout the paper, we denote by C and c absolute constants, possibly different. Occasionally, we shall number the constants for clarity.)

Lemma 1.4. *There exists a $\gamma > 0$ such that $\mathbb{P}(\omega_\varphi(\delta) > \delta^\gamma) \le C\delta^2$ $\forall \delta > 0$ (ω_φ is the modulus of continuity of φ).*

Proof. Applying Θ inductively we get

$$\rho_{2^{-(k+1)}}(x) = \tfrac{1}{k!}\log^k(x^{-1})\,. \tag{1}$$

So, for a given $b > 0$ we have

$$\mathbb{P}\big(\varphi(2^{-l}) > e^{-bl}\big) \le \frac{1}{(l-1)!}\int_{e^{-bl}}^1 \log^{l-1}\frac{1}{x}\,dx < \frac{(bl)^l}{(l-1)!} < (Cb)^l\,.$$

From the definition of $\mathbb{P}$, it follows that the increment of φ on any dyadic interval $[d2^{-l},(d+1)2^{-l}]$ has the same distribution as $\varphi(2^{-l})$; we obtain

$$\mathbb{P}\Big(\max_d\big[\varphi((d+1)2^{-l}) - \varphi(d2^{-l})\big] > e^{-bl}\Big) \le 2^l\mathbb{P}\big(\varphi(2^{-l}) > e^{-bl}\big) \le (Cb)^l\,,$$

and choosing b sufficiently small, we get the result.

Remarks. (i) It is possible to prove that $\gamma_0 = \inf\{\gamma|\varphi \in H^\gamma$ almost surely$\}$ is given as the smaller solution of the transcendental equation $\alpha(1/2)^\alpha(2e\log 2) = 1$. A calculation shows $\gamma_0 \approx 0.33$.

(ii) Using the inequality $\int_0^\varepsilon \log^l x^{-1}\,dx < C\varepsilon\log^l\varepsilon^{-1}$ we get from the proof above,

$$\mathbb{P}\Big\{\min_{0\le x\le 1}\big[\varphi(x+\delta) - \varphi(x)\big] < \delta^\beta\Big\} < \delta^2$$

for β sufficiently small. In particular, this means that φ^{-1} has almost surely a Hölder smoothness of some fixed order. Again, the exact value can be calculated – it is the inverse of the larger solution of the equation from (i) ≈ 0.26.

(iii) Let us consider the following variation on $\mathbb{P}$: Let $\mathbb{Q}_\varepsilon$ be a probability measure created by taking $X_{n,k}$ to be uniform variables on the interval $\left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right]$ (it is also a Dubins–Freedman measure). It is easy to see that a $\mathbb{Q}_\varepsilon$-random φ has a.s. a Hölder smoothness of $1-\delta$, where $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (and the same holds for φ^{-1}). We conjecture that our results (Theorems 2-6) hold also for $\mathbb{Q}_\varepsilon$, for any ε.

1.4. To further analyze the ρ_α's we need the following definition: a function f on (0,1) is "positive and then negative" if for some $x_0 \in (0,1)$ we have that $f(x) > 0$ for $x < x_0$ and $f(x) < 0$ for $x > x_0$.

LEMMA 1.5. *If $\alpha < \beta$ then $\rho_\alpha(x) - \rho_\beta(x)$ is positive and then negative. In particular, $\rho_\alpha(x) = \rho_\beta(x)$ at exactly one value of x.*

Proof. We already know (Theorem 1, iv) that $\rho_\alpha(x)$ is strictly decreasing for $\alpha < 1/2$ and strictly increasing for $\alpha > 1/2$. These two facts (remembering that $\rho_{1/2} = 1$) give us what we need for $\alpha \le 1/2 \le \beta$.

CLAIM. *If f and g are continuous on $(0,1)$, $f-g$ is positive and then negative and if $\int f = \int g$, then $\Theta f - \Theta g$ is positive and then negative.*

Clearly, $\Theta(f-g)$ is strictly decreasing and then increasing, and $\Theta(f-g)(1) = 0$. Since Θ preserves integrals, $\int_0^1 \Theta(f-g) = 0$ and these three facts combined give us that $\Theta(f-g)$ is positive and then negative. □

Using the claim, we get our result for all $\alpha \le 1/4 \le \beta < 1/2$. Reversing it (see Theorem 1, iii), we get the result for $1/2 < \alpha \le 3/4 \le \beta$. Repeating this process (applying Θ and reversing) over and over we prove our claim for all $\alpha < \beta$. □

LEMMA 1.6. $\rho_{1/3} \le 2$.

Proof. The equations $\rho_{1/3} = \Theta\rho_{2/3}$ and $\rho_{1/3}(x) = \rho_{2/3}(1-x)$ (which come from Theorem 1) can be combined to an integral functional equation for $\rho_{1/3}$. Let us define an operator Ξ by $(\Xi f)(x) = (\Theta f(1-x)) = \int_x^1 \frac{f(1-y)}{y}\,dy$. $\rho_{1/3}$ is a stable point of Ξ. On the other hand, the space of stable points of Ξ is one dimensional; assume f and g are two linearly-independent stable points of Ξ. Then some combination $af+bg$ does not have a constant sign. Since they are both continuous, we get that $\|\Theta(af+bg)\| < \|af+bg\|$ which implies that $\|\Xi(af+bg)\| < \|af+bg\|$ which is a contradiction. Now, $1-x$ is a solution for $\Xi f = f$ and thus $\rho_{1/3}(x) = 2(1-x)$.

An alternative proof for the uniqueness of the solution is the following: combining the first and second derivatives of the equation $\Xi f = f$ one can get the linear equation $xf'' + f' + \frac{1}{1-x}f = 0$. The other solution of the equation is $(1-x)(\log(1-x) - \log x) - 1$. Any solution for the equation $\Xi f = f$, however, must satisfy $f(1) = 0$, and thus the coefficient of this second solution must be 0.

Remark. It is not difficult to show, using different methods, that $\rho_\alpha(x)$ is bounded for $\alpha \in (1/4, 3/4)$. On the other hand, $\rho_{1/4}$ is unbounded (see equation (1) above).

The above discussion was aimed at proving the following lemma:

Lemma 1.7. *Suppose* $1/3 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = 2/3$. *Then*

$$\sum_{i=1}^{n-1} \int_0^1 \left|\rho_{\alpha_i}(x) - \rho_{\alpha_{i+1}}(x)\right| dx \le 4\,.$$

Proof. We may assume that for some k, $\alpha_k = 1/2$. We shall start by evaluating the sum up to $i = k-1$. We know that $\rho_{\alpha_i} - \rho_{\alpha_{i+1}} = 0$ at exactly one point in the interval (0,1). When $\alpha_{i+1} < 1/2$ we get that $\rho_{\alpha_i} - \rho_{\alpha_{i+1}} = \Theta(\rho_{2\alpha_i} - \rho_{2\alpha_{i+1}})$. Since $\rho_{2\alpha_i} - \rho_{2\alpha_{i+1}}$ is positive and then negative we have that $\rho_{\alpha_i} - \rho_{\alpha_{i+1}}$ is decreasing and then increasing. In particular, it is decreasing on its positive part. The last sentence clearly holds also for $\alpha_{i+1} = 1/2$. Moreover, we know that $(\rho_{\alpha_i} - \rho_{\alpha_{i+1}})(0) < \infty$, so clearly

$$\int |\rho_{\alpha_i} - \rho_{\alpha_{i+1}}| = 2\int (\rho_{\alpha_i} - \rho_{\alpha_{i+1}})^+ \le 2(\rho_{\alpha_i} - \rho_{\alpha_{i+1}})(0)\ .$$

Summing these results we obtain half of what we need,

$$\sum_{i<k} \int |\rho_{\alpha_i} - \rho_{\alpha_{i+1}}| \le 2\sum_{i<k}(\rho_{\alpha_i} - \rho_{\alpha_{i+1}})(0) = 2(\rho_{1/3} - \rho_{1/2})(0) = 2$$

(the last equality comes from Lemma 1.6). The sum over $i \ge k$ has the same estimate.

2 A Pointwise Estimate for the Fourier Partial Sums

From now on we shall identify the interval [0,1] with the circle $\mathbb{T}$. This identification will give us the probability space $(\Phi, \mathbb{P})$ of all homeomorphisms $\varphi : \mathbb{T} \to \mathbb{T}$ preserving the orientation and keeping 0 as a fixed point. We shall denote by D_n the Dirchlet kernel with period 1, i.e. $D_n = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}$.

The main result of this section is

Theorem 2. *There exists constants $c > 0$ and M_0 such that for any $n \geq 1$, $t \in \mathbb{T}$, $f \in C(\mathbb{T})$ and $M \geq M_0$,*

$$\mathbb{P}\Big(\int D_n(t-x) \cdot (f \circ \varphi)(x) dx > M\|f\| \Big) \leq e^{-e^{cM}} .$$

Proof. Throughout the proof we assume that f, n and t are fixed. Clearly, we may assume $\|f\| \leq 1$. We use the following notation:

$D(x) := D_n(t-x)$.

q is the integer satisfying $2^{q-1} \leq n < 2^q$.

An arc in $[0,1]$ is the image of an arc in $\mathbb{T}$, i.e. an interval or a set of the form $[0,x] \cup [y,1]$.

Let $B \subset [0,1]$ be an arc and let $K \in L^1(B)$. Then we define

$$\gamma_K(B) := \sup_{I \subseteq B} \Big| \int_I K \Big| ,$$

where the supremum is taken over all arcs contained in B. If $K = D$ we will just write $\gamma := \gamma_D$.

Actually, K will be some affine transformation of the Dirichlet kernel. In this case, because K flips signs, $\gamma_K(B)$ will be rather small when compared to $\int_B |K|$. Notice also the following geometric fact: if $|J| \geq 1/n$ then $\gamma(J) \approx \frac{C}{nd(t,J)+1}$, where d is the cyclic distance on $\mathbb{T}$.

Lemma 2.1. *Assume $K \in L^1([1/3, 2/3])$ and $f \in C([0,1])$, $\|f\| \leq 1$. Then*

$$\left| \mathbb{E} \int_{1/3}^{2/3} K \cdot (f \circ \varphi) \right| \leq C \gamma_K \left(\left[\frac{1}{3}, \frac{2}{3} \right] \right)$$

where C is some absolute constant.

Proof. Our idea is to perform integration by parts on $\int K \cdot (f \circ \varphi)$. Since φ is singular we cannot do it directly. We therefore approximate the integral with a finite sum and replace the integration by parts with Abel's formula. Let $1/3 = x_0 < x_1 < \cdots < x_l = 2/3$ and denote $\varepsilon_x := \max |x_{i+1} - x_i|$. Let us define $S_1 = \int_{1/3}^{2/3} K \cdot (f \circ \varphi)$ and $S_2 = \sum_{i=1}^{l} \big(\int_{x_{i-1}}^{x_i} K \big) \cdot f(\varphi(x_i))$. A rough estimation of the difference between the sum and the integral gives

$$|S_1 - S_2| \leq \Big(\int |K| \Big) \cdot \max_{\substack{\delta < \varepsilon_x \\ 1/3 \leq x \leq 2/3}} \big| f(\varphi(x+\delta)) - f(\varphi(x)) \big| .$$

Now let us choose arbitrarily small ε_y and δ and consider the event

$$X := \big\{ \varphi : \omega_\varphi(\varepsilon_x) > \varepsilon_y \big\} .$$

Lemma 1.4 will gives us that for ε_x suficiently small we shall have $\mathbb{P}(X) < \delta$. Further,

$$\begin{aligned}\mathbb{E}\big(|S_1 - S_2|\big) &= \mathbb{P}(X)\cdot\mathbb{E}\big(|S_1-S_2|\big|X\big) + \mathbb{P}(X^C)\cdot\mathbb{E}\big(|S_1-S_2|\big|X^C\big)\\ &\le 2\delta\int|K| + \mathbb{E}\big(|S_1-S_2|\big|X^C\big)\\ &\le 2\delta\int|K| + \Big(\int|K|\Big)\cdot\max_{\substack{\eta<\varepsilon_y\\ 0\le y\le 1}}\big|f(y+\eta)-f(y)\big|\\ &\le \Big(\int|K|\Big)\big(2\delta+\omega_f(\varepsilon_y)\big)\,.\end{aligned}$$

Therefore by choosing ε_x sufficiently small, we can make $\mathbb{E}(|S_1 - S_2|)$ arbitrarily small and refer to $\mathbb{E}(S_2)$ as an approximation of $\mathbb{E}(S_1)$.

Now let's apply Abel's transformation. We get

$$S_2 = \sum_{i=1}^{l-1}\bigg(\int_{1/3}^{x_i}K\bigg)\cdot\big(f(\varphi(x_i)) - f(\varphi(x_{i+1}))\big) + \bigg(\int_{1/3}^{2/3}K\bigg)\cdot f\big(\varphi(x_l)\big)$$

and thus

$$|\mathbb{E}S_2| \le \gamma_K\left(\left[\frac{1}{3},\frac{2}{3}\right]\right)\left(1+\sum_{i=1}^{l-1}\big|\mathbb{E}f(\varphi(x_i)) - \mathbb{E}f(\varphi(x_{i+1}))\big|\right).$$

Let us recall the definition of the density functions ρ_x of $\varphi(x)$ from §1. It is well known that

$$\mathbb{E}\big(f(\varphi(x))\big) = \int f(y)\cdot\rho_x(y)dy\,,$$

which gives us

$$\big|\mathbb{E}(f(\varphi(x_i))) - \mathbb{E}(f(\varphi(x_{i+1})))\big| = \bigg|\int f\cdot\rho_{x_i} - f\cdot\rho_{x_{i+1}}\bigg| \le \int|\rho_{x_i} - \rho_{x_{i+1}}|\,.$$

We apply Lemma 1.7 and we are done. The estimate $\sum_{i=1}^{l-1}\int|\rho_{x_i} - \rho_{x_{i+1}}| \le 4$ gives

$$\left|\mathbb{E}\sum_{i=1}^{l-1}\bigg(\int_{1/3}^{x_i}K\bigg)\cdot\big(f(\varphi(x_i)) - f(\varphi(x_{i+1}))\big)\right| \le 4\gamma_K\left(\left[\frac{1}{3},\frac{2}{3}\right]\right),$$

then $|\mathbb{E}S_2| \le 5\gamma_K\big(\big[\frac{1}{3},\frac{2}{3}\big]\big)$, and therefore $\big|\mathbb{E}\int_{1/3}^{2/3}K\cdot(f\circ\varphi)\big| \le 5\gamma_K\big(\big[\frac{1}{3},\frac{2}{3}\big]\big)$. □

LEMMA 2.2. *Let B be a dyadic interval. Let I be its middle third. Then*

$$\left|\mathbb{E}\bigg(\int_I D\cdot(f\circ\varphi)\;\bigg|\;\varphi(\partial B)\bigg)\right| \le C\gamma(I)\,,$$

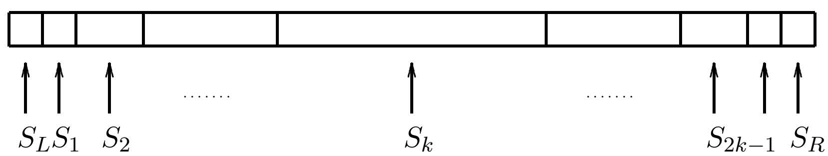
where C is some absolute constant.

Proof. We apply the previous lemma to $K = D \circ L$, where L is the affine increasing transformation carrying $[0,1] \to B$, with $f \circ L$ instead of f and use the scaling invariance of $\mathbb{P}$ on B. $\square$

LEMMA 2.3. *Let $B = [b_0, b_1]$ be a dyadic interval and assume $t \notin B$. Then*

$$\left|\mathbb{E}\left(\int_B D \cdot (f \circ \varphi) \mid \varphi(\partial B)\right)\right| \leq \gamma(B)\big(C|\log\gamma(B)| + C\big) .$$

Proof. If $|B| \leq 2^{-q}$ we have nothing to prove since D flips signs no more than once on B which implies that $\int_B |D| \leq 2\gamma(B)$ and we need no probabilistic considerations. Therefore, we can write $|B| = 2^{k-q}$. Divide B into $2k+1$ subblocks as in the following diagram:



S_k is the middle third, S_{k-1} is the middle third of the left half, S_{k-2} is the middle third of the left quarter, etc. S_L and S_R are the remainders and a short calculation shows that $|S_L| = |S_R| = \frac{2}{3}2^{-q}$. The same considerations as above show that $\int_{S_L} |D| \leq 2\gamma(B)$ which implies $\big|\mathbb{E}\int_{S_L} D \cdot (f \circ \varphi) \mid \varphi(\partial B)\big| \leq 2\gamma(B)$. The same holds for S_R. To estimate the integral for the S_j's we can use Lemma 2.2. For S_k we can use Lemma 2.2 with no modification and obtain $\big|\mathbb{E}\big(\int_{S_k} D \cdot (f \circ \varphi) \mid \varphi(\partial B)\big)\big| \leq C\gamma(S_k)$. For the other S_j's the proof is almost as simple: suppose for simplicity that $j \leq k-1$. Then

$$\begin{aligned}\mathbb{E}\left(\int_{S_j} D \cdot (f \circ \varphi) \mid \varphi(\partial B)\right) &= \mathbb{E}\mathbb{E}\left(\int_{S_j} D \cdot (f \circ \varphi) \mid \varphi(\partial B), \varphi(b_0 + 2^{j-q})\right) \\ &= \mathbb{E}\mathbb{E}\left(\int_{S_j} D \cdot (f \circ \varphi) \mid \varphi(b_0), \varphi(b_0 + 2^{j-q})\right),\end{aligned}$$

and we have the situation of Lemma 2.2, with B replaced by $[b_0, b_0 + 2^{j-q}]$ and I replaced by S_j – which implies

$$\leq \mathbb{E}\big(C\gamma(S_j)\big) = C\gamma(S_j)\,,$$

and so

$$\begin{aligned}\left|\mathbb{E}\left(\int_B D \cdot (f \circ \varphi) \,\middle|\, \varphi(\partial B)\right)\right| &\leq C\left(\gamma(S_L) + \gamma(S_R) + \sum \gamma(S_j)\right) \\ &\leq C\gamma(B) + C\sum \frac{1}{nd(t, S_j)}\,.\end{aligned}$$

Suppose now that t is closer or equal to the left side of B. We write $\sum_{j=1}^{k} = \sum' + \sum''$, where $\sum''$ is taken over the S_j's for which $d(t, S_j) \geq 2d(t, B)$ (if such j exist). Clearly, $\sum'$ holds no more than $\log_2(nd(t, B)) \leq C|\log \gamma(B)|$ summands, and for each one $\gamma(S) \leq \gamma(B)$ since $S \subset B$. On $\sum''$, $\frac{1}{d(t,S_j)}$ drop exponentially and then $\sum'' \leq C\frac{1}{nd(t,S_{j_0})} \leq C\gamma(S_{j_0}) \leq C\gamma(B)$ (j_0 is the first summand in $\sum''$). The same calculation holds for the case the $j \geq k$ if t is closer to the right side of B. Now, if $j \leq k$ and t is closer to the right side of B then we can simply argue that $d(t, S_j) > d(t, S_{2k-j})$ and thus $\sum_{j=1}^{k} \leq \sum_{j=k}^{2k-1}$ and we are done. □

Remark. The lemma also holds if $t \in B$, in which case it takes the form

$$\left|\mathbb{E}\left(\int_B D \cdot (f \circ \varphi) \;\Big|\; \varphi(\partial B)\right)\right| \leq C\,.$$

The proof is similar.

Lemma 2.4. *Let I be an arc. Let $B \subset I^C$ be a dyadic interval (I^C denotes the complement of I). Then*

$$\left|\mathbb{E}\left(\int_B D \cdot (f \circ \varphi) \;\Big|\; \varphi|_I\right)\right| \leq \gamma(B)\big(C|\log \gamma(B)| + C\big)\,.$$

Proof. It is clear from the construction of measure $\mathbb{P}$ that the information on $\varphi|_I$ adds nothing to $\varphi|_B$ when $\varphi|_{\partial B}$ is known. Thus this lemma is derived from Lemma 2.3 by integration over $\varphi|_{\partial B}$. □

Lemma 2.5. *Let I be an arc. Then*

$$\left|\mathbb{E}\left(\int_{I^C} D \cdot (f \circ \varphi) \;\Big|\; \varphi|_I\right)\right| \leq \gamma(I^C)\big(C\log^2 \gamma(I^C) + C\big)\,.$$

Proof. Following the proof of Lemma 2.3 we divide I^C into a finite number of disjoint intervals; two of them (those nearest I) of length $< C/n$, and all other dyadical of length $> 1/n$, at most two of each size. Denote them by B_j. For each B_j we use the estimate of Lemma 2.4 and we get

$$\left|\mathbb{E}\left(\int_{I^C} D \cdot (f \circ \varphi) \;\Big|\; \varphi|_I\right)\right| \leq 2\gamma(I^C) + \sum_{B_j} \gamma(B_j)\big(C|\log \gamma(B_j)| + C\big)\,.$$

As in Lemma 2.3, we get a sequence of intervals of length $C|\log \gamma(I^C)|$ for which we estimate $\gamma(B_j) \leq \gamma(I^C)$, and in the remaining intervals $\gamma(B_j)$ will drop exponentially. □

From now on, it will be more natural to change the notation somewhat. Let I_k be a symmetric arc centered at t which holds exactly $2k-1$ peaks of the Dirichlet kernel. Denote $Y_k := \int_{I_k^C} D\cdot(f\circ\varphi)$. Since $d(t, I_k^C) = k/n$ (d is the cyclic distance, as above), $\gamma(I_k^C) \approx Ck^{-1}$. Thus, Lemma 2.5 can be restricted to arcs centered at t, and reformulated as

LEMMA 2.5′. *For $k>1$, $|\mathbb{E}(Y_k \mid \varphi|_{I_k})| \le Ck^{-1}\log^2 k$.*

LEMMA 2.6. $\mathbb{E}(Y_k^2 \mid \varphi|_{I_k}) \le Ck^{-1}\log^2 k$.

Proof. Denote $J_k = I_k \setminus I_{k-1}$, $J_1 = I_1$ and $X_i = \int_{J_i} D\cdot(f\circ\varphi)$. Then $Y_k = \sum_{i\ge k+1} X_i$ and

$$Y_k^2 = 2\sum_{i\ge k+1} X_i Y_i + \sum_{i\ge k+1} X_i^2 .$$

Now, $\int_{J^i} |D| \le Ci^{-1}$ which gives the non-probabilistic estimate

$$\sum_{i\ge k+1} X_i^2 \le C\sum_{i\ge k+1} i^{-2} \le Ck^{-1} .$$

For the first summand, using $J_i \subset I_i$, we can write,

$$\begin{aligned}\big|\mathbb{E}(X_iY_i \mid \varphi|_{I_k})\big| &\le \mathbb{E}\big(|X_i|\cdot|\mathbb{E}(Y_i \mid \varphi|_{I_k\cup J_i})| \;\big|\; \varphi|_{I_k}\big)\\ &\le \mathbb{E}\big(|X_i|\cdot\mathbb{E}|\mathbb{E}(Y_i \mid \varphi|_{I_i})| \;\big|\; \varphi|_{I_k}\big)\\ &\le \mathbb{E}\big(|X_i|\cdot\mathbb{E}(Ci^{-1}\log^2 i) \;\big|\; \varphi|_{I_k}\big)\\ &= Ci^{-1}\log^2 i\cdot\mathbb{E}\big(|X_i| \;\big|\; \varphi|_{I_k}\big) \le Ci^{-2}\log^2 i .\end{aligned}$$

This gives us

$$E\Bigg(\sum_{i\ge k+1} X_iY_i \;\Bigg|\; \varphi|_{I_k^C}\Bigg) \le \sum_{i\ge k+1} Ci^{-2}\log^2 i \le Ck^{-1}\log^2 k ,$$

which proves our lemma. □

REMARKS. (i) Using a slightly more sophisticated decomposition into dyadic blocks, one can remove the $\log^2 k$ factor from the above estimation. However, this gives no improvement to the final result.

(ii) In §4 we will need a version of Lemma 2.6 which is not restricted to arcs centered at t. The exact formulation is: For any arc $J\subset[0,1]$,

$$\mathbb{E}\Bigg(\bigg|\int_J D\cdot(f\circ\varphi)\bigg|^2 \;\Bigg|\; \varphi|_{J^C}\Bigg) < C\gamma(J)\big(\log^2\gamma(J)+1\big) .$$

The proof is identical to the proof of Lemma 2.6.

The last lemma enables us to complete the proof of Theorem 2 by a suitable modification of a classical independent-variable sum technique, which one can find in [K2, ch. 2].

Let C_1 satisfy that $\int_{I_j\setminus I_{j-1}}|D| < C_1/j$. For a given M, we denote by $j = j(M)$ the maximal integer such that $\int_{I_j}|D| \le M-1$. Clearly, there exists an absolute constant $c>0$ such that $j \ge e^{cM}$. We have,

$$\mathbb{P}\Big(\Big|\int_{\mathbb{T}} D\cdot(f\circ\varphi)\Big| > M\Big) \le \mathbb{P}\big(|Y_j|>1\big)\,.$$

Using Lemma 2.6, we can define $\mu := C\frac{\log j}{\sqrt{j}}$ which will satisfy

$$\mathbb{P}\big(|Y_s|>\mu \mid \varphi|_{I_s}\big) \le \tfrac14 \qquad \forall\, s\ge j\,. \tag{1}$$

Now, if M is sufficiently large ($M>M_0$), we may also assume that

$$\frac{C_1}{j} < \mu := C\frac{\log j}{\sqrt{j}}\,. \tag{2}$$

Lemma 2.7. *If, for a given $\varepsilon>0$ and $\nu\ge 1$, the inequality*

$$\mathbb{P}\big(|Y_s|>\nu\mu \bigm| \varphi|_{I_s}\big) \le \varepsilon \qquad \forall\, s\ge j$$

is true, then

$$\mathbb{P}\Big(\max_{r\ge s}|Y_r-Y_s| > (\nu+1)\mu \bigm| \varphi|_{I_s}\Big) \le \tfrac43\varepsilon \qquad \forall\, s\ge j\,.$$

Proof. We fix s and divide the event $B=\{\max_{r>s}|Y_r-Y_s|>(\nu+1)\nu\}$ into disjoint events

$$B_r = \big\{|Y_r-Y_s|>(\nu+1)\mu\,,\ |Y_l-Y_s|\le(\nu+1)\mu\,\forall\, l=s,\dots,r-1\big\} \qquad (r>s)\,.$$

Obviously, $B_r\cap\{|Y_r|\le\mu\}\subset\{|Y_s|>\nu\mu\}$. This gives us

$$\begin{aligned}\varepsilon &\ge \mathbb{P}\big(|Y_s|>\nu\mu \bigm| \varphi|_{I_s}\big) \ge \mathbb{P}\big(B\cap\{|Y_s|>\nu\mu\} \bigm| \varphi|_{I_s}\big)\\ &= \sum_{r>s}\mathbb{P}\big(B_r\cap\{|Y_s|>\nu\mu\} \bigm| \varphi|_{I_s}\big) \ge \sum_{r>s}\mathbb{P}\big(B_r\cap\{|Y_r|\le\mu\} \bigm| \varphi|_{I_s}\big)\,.\end{aligned}$$

Clearly, B_r belongs to the σ-algebra generated by $\varphi|_{I_r}$ and so (1) with r instead of s gives:

$$\mathbb{P}\big(B_r\cap\{|Y_r|\le\mu\} \bigm| \varphi|_{I_s}\big) \ge \tfrac34 P\big(B_r \bigm| \varphi|_{I_s}\big)$$

and thus $\varepsilon \ge \frac34 P(B\mid\varphi|_{I_s})$, which is what we need. □

Lemma 2.8. *With the assumptions of Lemma 2.7,*

$$\mathbb{P}\big(|Y_s|>(2\nu+2)\mu \bigm| \varphi|_{I_s}\big) \le \tfrac43\varepsilon^2\,.$$

Proof. Let us retain the definitions of B and B_r from the previous lemma. Notice that the values of $|Y_r|$ change by no more than $C_1 r^{-1} \le \mu$ at each stage. Therefore $|Y_s - Y_r|$ cannot rise above $(\nu+2)\mu$ without first being inside the interval $[(\nu+1)\mu, (\nu+2)\mu)$. Thus

$$\begin{aligned}\mathbb{P}\big(|Y_s| > (2\nu+2)\mu \bigm| \varphi|_{I_s}\big) &\le \sum_{r>s} \mathbb{P}\big(B_r \cap \{|Y_r| > \nu\mu\} \bigm| \varphi|_{I_s}\big) \\ &\le \sum_{r>s} \varepsilon \mathbb{P}(B_r \mid \varphi_{I_s}) = \varepsilon \mathbb{P}(B \mid \varphi_{I_s}) \le \tfrac{4}{3}\varepsilon^2\end{aligned}$$

(using Lemma 2.7 for the last estimate). □

Starting from the definition of μ we apply Lemma 2.8 inductively l times and get that

$$\mathbb{P}\big(|Y_s| > \mu d_l \bigm| \varphi|_{I_s}\big) \le \left(\tfrac{4}{3}\right)^{2^l - 1} \left(\tfrac{1}{4}\right)^{2^l} < \left(\tfrac{1}{3}\right)^{2^l},$$

where the d_l's are defined recursively by $d_1 = 1$, $d_l = 2d_{l-1} + 2$. Clearly $d_l \le C2^l$. Picking a maximal l such that $\mu d_l < 1$ we get that $l > cM$ and thus $\mathbb{P}(|Y_s| > 1 \mid \varphi|_{I_s}) \le (1/3)^{2^l} \le e^{-e^{cM}}$. In particular it holds for $s = j$ and the theorem is proved. □

Remark. The theorem also holds for L^∞ functions. To prove it one only needs to use Lusin's "correction" Theorem and estimate the remainder by Lemma 1.3.

3 Global Estimates

For a given f we denote by $S_n(f)$ the n-th partial sum of its Fourier expansion. By $\|\cdot\|$ we always mean the norm in $C(\mathbb{T})$.

Theorem 3. *For any $f \in C(\mathbb{T})$, $\|S_n(f \circ \varphi)\| = o(\log\log n)$ almost surely.*

Theorem 4. *If $\omega_f(\delta) = o(\log\log 1/\delta)^{-1}$ then $f \circ \varphi \in U(\mathbb{T})$ almost surely.*

Remember that $U(\mathbb{T})$ is the class of all functions on $\mathbb{T}$ having a uniformly converging Fourier series. We will see in §4 that both results are sharp.

We need the following lemma which is an easy corollary to Theorem 2.

Lemma 3.1. *If $\|f\| \le 1$ then $\mathbb{P}(\|S_n(f \circ \varphi)\| > C_1 \log\log n) \le n^{-2}$ for $n > C_2$.*

Proof. For any $t \in \mathbb{T}$, Theorem 2 gives $\mathbb{P}(|S_n(f\circ\varphi)(t)| > C_1 \log\log n) < n^{-4}$ for $n > C_2$. It follows that for a fixed $n > C_2$ with probability $> 1 - n^{-2}$, we have

$$\big|S_n(f\circ\varphi)(dn^{-2})\big| \le C_1 \log\log n\,, \quad \forall\, 1 \le d \le n^2\,.$$

But $S_n(f\circ\varphi)$ is a trigonometric polynomial of degree n with the modulus of each coefficient ≤ 1, so clearly $\left\|\frac{d}{dt}S_n(f\circ\varphi)(t)\right\|\le Cn^2$ and we get from the previous estimate that $\|S_n(f\circ\varphi)\| < C_1\log\log n + C$ with probability $> 1-n^{-2}$. □

Proof of Theorem 3. Let us fix an arbitrary $\varepsilon>0$ and find a C^1 function g such that $\|f-g\|<\varepsilon$. It is well known that there is a constant $C=C(g)$ for which $\|S_n(g\circ\varphi)\|\le C(g)$ uniformly for $\varphi\in\Phi$. But the previous lemma gives

$$\sup_{n\ge N>C_2}\frac{1}{\log\log n}\left\|S_n((f-g)\circ\varphi)\right\|<\varepsilon C$$

with probability $>1-CN^{-1}$ and the theorem follows. □

Remark. If $f\in L^\infty(\mathbb{T})$ then using the Remark to Theorem 2, one gets the same estimate, with o replaced by O.

Corollary. *For any $f\in C(\mathbb{T})$, there exists φ such that φ and φ^{-1} are Hölder and $\|S_n(f\circ\varphi)\|=o(\log\log n)$.*

We do not know whether it is possible to improve this result in either of the following two directions: replacing bi-Hölder by bi-Lipschitz; and replacing $o(\log\log n)$ with $O(1)$.

Proof of Theorem 4. As usual, we assume that $\|f\|\le 1$. Using Lemma 1.4 we fix $\gamma>0$ such that for every n, $\omega_\varphi(n^{-1})\le n^{-\gamma}$ with probability $>1-Cn^{-2}$. For a given n sufficiently large we pick N, $n^{\gamma/2}\le N\le 2n^{\gamma/2}$, and define $g\in C(\mathbb{T})$ to satisfy $g(d/N)=f(d/N)$, $1\le d\le N$, and linear between any two neighbouring points. Clearly, $\|g'\|\le 2N$ so

$$\omega_\varphi(n^{-1})\le n^{-\gamma}\Rightarrow\omega_{g\circ\varphi}(n^{-1})<2Nn^{-\gamma}<4n^{-\gamma/2}\,.$$

Now using the standard estimate

$$\left\|S_n(F)-F\right\|\le C\log n\cdot\omega_F(n^{-1})\qquad\forall\, F\in C(\mathbb{T})\,,$$

we obtain $\|S_n(g\circ\varphi)-g\circ\varphi\|<Cn^{-\gamma/2}\log n$ with probability $>1-Cn^2$. On the other hand, Lemma 3.1 gives

$$\left\|S_n((f-g)\circ\varphi)\right\|<C\|f-g\|\log\log n<C\omega_f(N^{-1})\log\log n$$

with probability $>1-n^{-2}$, so for almost every $\varphi\in\Phi$

$$\left\|S_n(f\circ\varphi)-f\circ\varphi\right\|<C\omega_f(N^{-1})\log\log n+Cn^{-\gamma/2}\log n+C\omega_f(N^{-1})$$

for every $n>N(\varphi)$, and the result follows. □

Remark. If $\omega_f(\delta)=O(\log\log 1/\delta)^{-1}$ then $f\circ\varphi$ has a.s. uniformly bounded Fourier partial sums.

4 Examples of Divergence

The aim of this section is to show that the main estimates of section 3 cannot be improved.

Theorem 5. *For every $s(n) = o(\log\log n)$ there exists an $f \in C(\mathbb{T})$ such that*

$$\limsup \frac{1}{s(n)} \big\| S_n(f \circ \varphi) \big\| = \infty \text{ a.s.}$$

Theorem 6. *There exists a function f with $\omega_f(\delta) = O(\log\log 1/\delta)^{-1}$ such that $f \circ \varphi \notin U(\mathbb{T})$ a.s.*

4.1. We start with a few lemmas.

LEMMA 4.1. *Let φ_0 be a Lipschitz homeomorphism of $[0,1]$ with a constant K, i.e. $|\varphi_0(x) - \varphi_0(y)| \le K|x-y|$. Then*

$$\mathbb{P}\big\{ \|\varphi - \varphi_0\| < r \big\} > c(K)^{1/r}, \ \forall\, r > 0$$

where $c(K)$ is some positive constant.

Proof. It is clearly enough to consider $r = \frac{K+2}{2^q}$, where q is an integer. For $s \le q$ we denote

$$X_s := \big\{ \varphi : |\varphi(j2^{-s}) - \varphi_0(j2^{-s})| < 2^{-q}, \ \forall\, 0 < j < 2^s \big\}.$$

Let us estimate $\mathbb{P}(X_{s+1}|X_s)$. If $\varphi \in X_s$ with some $s < q$ then for any odd j, $0 < j < 2^{s+1}$, the probability of the event

$$\big| \varphi(j2^{-(s+1)}) - \varphi_0(j2^{-(s+1)}) \big| < 2^{-q}$$

can be estimated from below by

$$\frac{2^{-q}}{\left| \varphi\left(\frac{j-1}{2^{s+1}}\right) - \varphi\left(\frac{j+1}{2^{s+1}}\right) \right|} \ge \frac{2^{-q}}{K2^{-s} + 2^{-q+1}},$$

and for different j's these event are independent. So $\mathbb{P}(X_{s+1}|X_s) > (K2^{q-s} + 2)^{-2^s}$ which implies

$$\mathbb{P}(X_q) = \prod_{s=0}^{q-1} \mathbb{P}(X_{s+1} \mid X_s) > \prod_{s=0}^{q-1} (K+2)^{-2^s} 2^{-(q-s)2^s}$$

$$> \exp -\left(2^q \log(K+2) + 2^q \log 2 \sum_{j=1}^{\infty} \frac{j}{2^j} \right) = \exp\left(-\frac{C(K)}{r} \right).$$

To finish the proof we only need to notice that X_q implies $\|\varphi - \varphi_0\| < r$. □

LEMMA 4.2. *Let N be a natural number, I a dyadical interval in $(0,1)$, t the middle of I, and suppose we fix $\varphi|_{\partial I}$. Set $l = N|\varphi I|$ and assume l is large enough ($l > l_0$). Then there exists a number n, $n|I|/l \in (1/2, 1)$ such that*

$$\mathbb{P}\left\{ \left| \int_I D_n(t-x) \sin \pi N \varphi(x) dx \right| > c \log l \;\middle|\; \varphi|_{\partial I} \right\} > c^l \,,$$

where c is some absolute positive constant.

Proof. The properties of the Dirichlet kernel imply immediately that there exists an n as above and a piece-linear homeomorphism $\psi_0 = \psi_0(I, n, N) : I \to \varphi(I)$ which will satisfy

(i) $|\psi_0'| < 10|\varphi(I)|/|I|$
(ii) ψ_0 has exactly two inflection points in $I' = \left[t - \frac{1}{4}|I|, t + \frac{1}{4}|I|\right]$, both not more than $1/n$ away from t.
(iii) $\operatorname{sign} \sin \pi N \psi_0(x) = \operatorname{sign} D_n(t-x)$ on the interval I'.

These properties imply

$$\left| \int_I D_n(t-x) \sin \pi N \psi_0(x) dx \right| > c_1 \int_{I'} |D| - c > c_2 \log l - c \,. \tag{1}$$

Now let ψ be any homeomorphism $\psi : I \to \varphi(I)$ with the condition

$$\|\psi - \psi_0\| < \frac{c_1 |\varphi(I)|}{20 l} \,. \tag{2}$$

Then clearly,

$$\left| \int_I D \cdot \big(\sin \pi N \psi(x) - \sin \pi N \psi_0(x) \big) dx \right| < \int_I |D| \cdot \pi N \cdot \frac{c_1 |\varphi(I)|}{20 l} < \frac{1}{2} c_2 \log l$$

and we get (for ψ) the same inequality as (1) but with a different constant. Let us now estimate the probability of the event (2). Using the scaling invariance of $\mathbb{P}$ we apply Lemma 4.1 to a scaled version of ψ_0, $r = c_1/20l$ and get the result of the lemma. □

LEMMA 4.3. *For a given $\varepsilon > 0$ and N sufficiently large $N > N_0(\varepsilon)$ one can find with probability $> 1 - \varepsilon$ a random number n, a random dyadical interval I and a random point $t \in I$ such that the following conditions hold:*

(i) $\left| \int_I D_n(x-t) \sin \pi N \varphi(x) dx \right| > c \log \log N$;
(ii) $N^c < n < N^C$;
(iii) $c \log N < n|I| < C \log N$.

Proof. First, using Lemma 1.4 and remark (ii) that follows, we can find a $d > 1$ such that

$$\mathbb{P}\left\{ h^d < \max |\varphi(x) - \varphi(x+h)| < h^{1/d} \;\forall\, 0 < h < h_\varepsilon \right\} > 1 - \tfrac{\varepsilon}{3} \,. \tag{3}$$

For any N satisfying $\frac{1}{2}N^{1/2d} > h_\varepsilon^{-1}$ we define Q to be the power of 2 which satisfies

$$\tfrac{1}{h_\varepsilon} < \tfrac{1}{2}N^{1/2d} < Q < N^{1/2d}\,. \tag{4}$$

Fix now a dyadical interval $J = \left[\frac{j-1}{Q}, \frac{j}{Q}\right]$ and suppose that

$$\left|\varphi(J)\right| > Q^{-d}\ (> N^{-1/2})\,, \tag{5}$$

then obviously, assuming that N is sufficiently large, one can always select a random dyadical subinterval $I \subset J$ such that

$$c_1\frac{\log N}{N} < \left|\varphi(I)\right| < 2c_1\frac{\log N}{N}\,, \tag{6}$$

where c_1 is some constant that will be fixed later. To pick such an I, we can start with $I_0 := J$ and always choose I_n to be the larger half of I_{n-1} until, for some n, $|I_n|$ will satisfy (6) and then define $I := I_n$. This selection process has the important property that if $\varphi' = \varphi$ outside of $I = I(\varphi)$ then $I(\varphi') = I(\varphi)$. Thus we may apply Lemma 4.2 to I and get

$$\mathbb{P}\left\{ \sup \left| \int_I D_n(x-t)\sin \pi N\varphi(x)dx \right| > c\log(c_1 \log N) \;\Big|\; \varphi(\partial I) \right\} > c^{2c_1 \log N}$$

where the supremum is taken over all $t \in \mathbb{T}$, dyadical $I \subset J$ and n, satisfying conditions (6) and (7)

$$\frac{c_1 \log N}{2|I|} < n < \frac{c_1 \log N}{|I|}\,. \tag{7}$$

Assuming that (5) holds for all J's and using independence, we get the estimate for the conditional probability,

$$\mathbb{P}\left\{ \sup_{t,I,n} \left| \int_I D_n(x-t)\sin \pi N\varphi(x)dx \right| > c\log\log N \;\Big|\; (5)\text{ holds } \forall j \right\}$$
$$> 1 - (1 - c^{2c_1 \log N})^Q\,,$$

and, if c_1 is chosen sufficiently small,

$$> 1 - \tfrac{\varepsilon}{3}\,.$$

The supremum is taken as above, for I, t and n satisfying (6) and (7) but over all dyadical $I \subset [0,1]$.

Because of (3), assumption (5) occurs for all j with probability $> 1-\frac{1}{3}\varepsilon$ and thus (i) and (iii) are fulfilled with probability $> \left(1-\frac{1}{3}\varepsilon\right)^2$ (see condition (7)). We lose another $\frac{1}{3}\varepsilon$ to get (ii) from (3), (6) and (7). □

4.2. Our next step might be of independent interest.

Lemma 4.4.
$$\mathbb{P}\Big(\Big|\int_J D_n(t-x)\cdot(f\circ\varphi)\Big| > M \;\Big|\; \varphi|_{J^C}\Big) < \frac{C}{M}$$
where C is an absolute constant, independent of n, t and J.

We refer the reader to remark (ii) following the proof of Lemma 2.6. Lemma 4.4 is an easy consequence of it.

Remark. It is possible to get a double-exponential estimation of the probability in Lemma 4.4, as in Theorem 1, but we do not need it here.

Lemma 4.5. *Let $\|f\| < 1$, $M > 1$, r and $n_0 < n_1$ be given with the condition*
$$\mathbb{P}\Big\{\sup_{\substack{t\in\mathbb{T}\\ y\in[0,1]\\ n\in[n_0,n_1]}} \Big|\int_{[0,y]} D_n(t-x)f\big(\varphi(x)\big)dx\Big| > M\Big\} > r\,.$$
Then
$$\mathbb{P}\Big\{\sup_{n\in[n_0,n_1]} \big\|S_n(f\circ\varphi)\big\| > \tfrac{M}{2}\Big\} > r - \tfrac{C}{M}\,.$$

Proof. Denote by B the event above, so $\mathbb{P}(B) > r$. On the set of all triplets $\alpha = \{y,t,n\}$ we introduce an order

$\alpha_1 \prec \alpha_2 \Leftrightarrow y_1 < y_2$ or $y_1 = y_2$ and $t_1 < t_2$ or $y_1 = y_2,\ t_1 = t_2$ and $n_1 < n_2$.

Let $\alpha(\varphi)$ be the minimal α for which $\big|\int_0^y D_n\cdot(f\circ\varphi)\big| > M$. This allows us to partition B into disjoint events $B^\alpha = \{\varphi\in B \mid \alpha(\varphi) = \alpha\}$. One can see that every B^α depends only on $\varphi|_{[0,y(\alpha)]}$. We now fix $\alpha = \{y,t,n\}$. Then using Lemma 4.4 with $J = [y,1]$ we get
$$\mathbb{P}\Big\{\big|S_n(f\circ\varphi)(t)\big| < \tfrac{M}{2} \mid B^\alpha\Big\} < \tfrac{C}{M}$$
and after integration on α,
$$\mathbb{P}\Big\{\max_{n_1<n<n_2} \big\|S_n(f\circ\varphi)\big\| < \tfrac{M}{2} \mid B\Big\} < \tfrac{C}{M}\,,$$
and the lemma follows. □

We mention a corollary. Denote by U_0 the set of continuous functions with uniformly bounded partial sums, and by $\widetilde{U}$ the set of f for which
$$\Big|\int_I D_n(x-t)f(x)dx\Big| \le K(f)$$
for all $t\in\mathbb{T}$, intervals $I\subset\mathbb{T}$ and $n\in\mathbb{N}$. Clearly, $\widetilde{U}\subset U_0$ and it is well known that the imbedding is proper. However, from a probabilistic point of view these two classes are identical.

COROLLARY. *For any* $f \in C(\mathbb{T})$, $\mathbb{P}(f \circ \varphi \in U_0 \backslash \widetilde{U}) = 0$.

Proof. Notice first that $\widetilde{U} = \{f : |\int_0^y D \cdot f| \le K(f)\}$, i.e. that we can restrict our attention to intervals of the type $[0, y]$. The corollary then follows from Lemma 4.5 by taking a limit as $M \to \infty$. □

COROLLARY. *If* $f \in C(\mathbb{T})$ *and* ψ *any piecewise linear homeomorphism of* $\mathbb{T}$ *then* $f \circ \varphi \in U(\mathbb{T}) \Leftrightarrow f \circ \varphi \circ \psi \in U(\mathbb{T})$ *almost surely.*

The last corollary is obviously closely connected with a zero-one law for the property $f \circ \varphi \in U(\mathbb{T})$. However, the naive proof of the zero-one law requires us to compare $f \circ \varphi$ with $f \circ \psi \circ \varphi$. A correct proof of the zero-one law therefore requires an additional component, which we shall not describe here.

4.3. We are now able to finish the proofs of Theorems 5 and 6.

Proof of Theorem 5. Clearly, we may assume that $s(n)$ is increasing. Define by induction a fast increasing sequence N_k and a fast decreasing sequence b_k with the conditions

(i) $\sum_{q<k} b_q N_q = o(b_k \log\log N_k)$
(ii) $\sum_{q>k} b_q = o(b_k / \log N_k)$
(iii) $s(N_k^c) = o(b_k \log\log N_k)$ where c is taken from Lemma 4.3, (ii).

(At each step, choose N_k sufficiently large such that (i) and (iii) are fulfilled, and then choose b_{k+1} such that all conditions (ii), for all k, will be fulfilled). With these constants, we may define

$$f(x) := \sum b_k \sin \pi N_k x .$$

Let $n \in [N_k^c, N_k^C]$, c and C from Lemma 4.3. To estimate $S_n(f \circ \varphi)$ we shall define $f_1 := \sum_{q<k} b_q \sin \pi N_q x$ and $f_2 := \sum_{q>k}$. The standard estimate $S_n(g) < C\mathrm{V}(g)$ where $\mathrm{V}(g)$ is the variation of g will give

$$\|S_n(f_1 \circ \varphi)\| < C \sum_{q<k} b_q N_q = o(b_k \log\log N_k)$$

uniformly for all φ and n as above. For f_2 we get

$$\|S_n(f_2 \circ \varphi)\| = O(\|f_2\| \log n) = o(b_k) .$$

Now, applying Lemmas 4.3 and 4.5 we get

$$\max_{n \in [N_q^c, N_q^C]} \|S_n(b_k \sin \pi N_k \varphi(x))\| > c b_k \log\log N_k ,$$

with probability $> 1 - o(1)$. As $s(n) < s(N_k^c) = o(b_k \log\log N_k)$ the theorem is proved. □

Proof of Theorem 6. Set

$$f(x) = \sum \frac{1}{\log\log N_k} \sin \pi N_k x \,,$$

where N_k is increasing sufficiently fast. One can easily see that f has the desired $\omega_f(\delta)$.

To prove the divergence of $S_n(f)$ we write, as in the proof of Theorem 5,

$$f = f_1 + \frac{1}{\log\log N_k} \sin \pi N_k x + f_2 \,.$$

Standard estimates for the rate of convergence of Fourier partial sums of Hölder functions give us that, if N_k is increasing sufficiently rapidly, then

$$\max_{n\in[N_k^c, N_k^C]} \big\| S_n(f_1 \circ \varphi) - f_1 \circ \varphi \big\| = o(1) \,,$$

with probability $> 1 - o(1)$. For f_2, we may argue as in the proof of Theorem 5, and for the main term (denote it by f_3), Lemmas 4.3 and 4.5 give us

$$\max_{n\in[N_k^c, N_k^C]} \big\| S_n(f_3) - f_3 \big\| > c$$

and the theorem is proved. □

4.4 Remarks. i) Analyzing the proof of the last theorem one can see that in fact one can get not only divergence in C but also pointwise divergence at a random point (actually on a random dense set).

ii) On the other hand, one can localize our construction and get an example of an f with the same smoothness for which $f \circ \varphi$ has a Fourier series which converges pointwise everywhere, but not in $C(\mathbb{T})$. The localized function looks like $\frac{1}{n}\sin(e^{e^n}x + \psi_n)$ over the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ where the ψ_n are chosen to make f continuous.

It is interesting to compare this result with the Billard theorem, which states that for random functions of the type $\sum \pm a_n e^{inx}$ almost-sure-pointwise-convergence-everywhere and almost-sure-uniform-convergence are equivalent.

iii) A different variant on the pointwise question is the following: when is it true for a function f to satisfy that $f \circ \varphi$ has an a.s. convergent Fourier series at x_0 assuming that $\varphi(x_0) = y_0$? We give without proof the following result for the case $x_0 = y_0 = 0$: a sufficient condition for this behaviour is $|f(\delta) - f(0)| = o(\log\log\log 1/\delta)^{-1}$ and this condition is sharp.

iv) Using similar techniques, one can show that the remarks following Theorems 3 and 4 are also sharp.

5 Baire Categories

Here we prove that if, instead of the probabilistic approach above, one considers the "*typical*" homeomorphism φ in the sense of Baire categories then the situation completely changes: even the Dini–Lipshitz condition (0.2) does not imply the convergence of the Fourier series of $f \circ \varphi$.

We start with

Proposition. $\sum \omega_f(1/k)/k < \infty$ *implies that for every homeomorphism* $\varphi : \mathbb{T} \to \mathbb{T}$, $f \circ \varphi \in U(\mathbb{T})$.

The proof is based on a theorem of Baernstein and Waterman: $f \circ \varphi \in U \ \forall \varphi$ iff

$$\lambda(\delta) := \sup_{|x_1 - x_{2k}| < \delta} \left| \sum_{j=1}^{k} \frac{1}{j} \big(f(x_{2j}) - f(x_{2j-1}) \big) \right| \to 0 \qquad (\delta \to 0) \tag{1}$$

(here $\{x_j\}$ $(1 \le j \le 2k)$ is any monotone sequence on $[0,1]$), see [BW].

Now for a given such sequence, $|x_1 - x_{2k}| < \delta$, we have

$$\left| \sum_{j=1}^{k} \frac{1}{j} \big(f(x_{2j}) - f(x_{2j-1}) \big) \right| \le \sum \frac{1}{j} \big| f(x_{2j}) - f(x_{2j-1}) \big|$$
$$\le \sum \tfrac{1}{j} \omega_f \big(|x_{2j} - x_{2j-1}| \big) \le \sum \tfrac{1}{j} \omega_f(\delta_j) \,,$$

where δ_j $(1 \le j < k)$ is the decreasing rearrangement of the sequence $\{|x_{2j} - x_{2j-1}|\}$. Obviously $\delta_j \le \delta/j$, so we get (with any ν)

$$\lambda(\delta) \le \sum \frac{1}{j} \omega_f \left(\frac{\delta}{j} \right) \le \omega_f(\delta) \cdot \sum_{j \le \nu} \frac{1}{j} + \sum_{j > \nu} \frac{\omega_f(1/j)}{j} \,.$$

Now by choosing ν we can make the second term arbitrarily small and (1) follows.

Our aim now is to prove that the result is sharp even for a typical (instead of every) φ.

Let $\bar{\Phi}$ be the set of all nondecreasing continuous functions φ on [0,1], $\varphi(0) = 0, \varphi(1) = 1$ with the supremum norm. It is a complete metric space and one can see easily that the set Φ defined in 1.1 is a residual set in $\bar{\Phi}$ (as usual, a residual set is a set whose complement is of the first Baire category). Therefore it is a second category space and it makes sense to speak about "typical" elements of Φ.

Theorem 7. *Let* $\omega(\delta)$ *be a modulus of continuity satisfying the condition*

$$\sum \tfrac{1}{k} \omega \left(\tfrac{1}{k} \right) = \infty \,. \tag{2}$$

Then there exists a function $f \in H^\omega$ such that for every $t \in \mathbb{T}$

$$\limsup \left|S_n(f\circ\varphi;t)\right| = \infty\,, \tag{3}$$

for every $\varphi \in \Phi$ except for a set of the first category.

Proof. 1. Let f be a function $\in C(\mathbb{T})$, $\|f\|=1$ such that

$$\lambda(f;I) := \sup_{\substack{\{x_j\}\subset I\\ x_1>x_2>\cdots>x_{2k}}} \left|\sum_{j=1}^{k}\frac{1}{j}\big(f(x_{2j})-f(x_{2j-1})\big)\right| = \infty\,, \tag{4}$$

where I is any interval $\subset [0,1]$. Then for any $t \in \mathbb{T}$, (3) holds for typical φ. Indeed, fix some $0<t<1$, a $\varphi \in \Phi$ and $\varepsilon > 0$. Find $\delta > 0$ with the condition $\varphi(t-\delta) > \varphi(t)-\varepsilon$. For a given M, using (4), choose a $\widetilde{\varphi}$ on $[t-\delta,t]$, agreeing with φ on the boundries, such that for some n,

$$\left|\int_{t-\delta}^{t}(f\circ\widetilde{\varphi})(x)D_n(t-x)dx\right| > M+\tfrac{C_1}{\delta}+C_2\,,$$

where C_1 and C_2 are some absolute constants that will be fixed later. Now define $\widetilde{\varphi}$ to be a constant on $\left[t,t+\frac{1}{2}\delta\right]$ and we will get that for any n,

$$\left|\int_{t}^{t+\frac{1}{2}\delta}(f\circ\widetilde{\varphi})(x)D_n(t-x)dx\right| \le C\,.$$

Fix C_2 to be this C. Set $\widetilde{\varphi}$ to be equal to φ outside $[t-\delta,t+\delta]$ and interpolate $\widetilde{\varphi}$ linearly on $\left[t+\frac{1}{2}\delta,t+\delta\right]$. We will get, simply because $|f\circ\widetilde{\varphi}|\le 1$, that

$$\left|\int_0^{t-\delta}+\int_{t+\frac{1}{2}\delta}^{1}(f\circ\widetilde{\varphi})(x)D_n(t-x)dx\right| \le \frac{C}{\delta}\,.$$

Again, fix C_1 to be this C. This will give us

$$\left|S_n(f\circ\widetilde{\varphi};t)\right| > M\,.$$

It follows that the set $\{\varphi : \sup_n|S_n(f\circ\widetilde{\varphi};t)|>M\}$ is dense and certainly open in Φ, and we get (3) on a residual set of homeomorphisms.

2. Now we need only construct a function $f \in H^\omega$ satisfying (4). Denote

$$f_s(x) = \begin{cases}\omega(4^{-l})\sin 4^l\pi x & x\in(2^{-l},2^{-(l-1)}],\quad l<s\\ 0 & x\le 2^{-s}\end{cases}$$

and extend it periodically with period 1 on $\mathbb{R}$.

Clearly $\omega_{f_s}(\delta) < C\omega(\delta)$, $\delta>0$. Denote $x_j^{(s)}$ as the set of all points of local extremum of f_s on [0,1] in decreasing order. Then we have

$$\sum \tfrac{1}{j}\left|f_s(x_{j+1}^{(s)})-f_s(x_j^{(s)})\right| > c\sum_{\ell<s}\omega(4^{-\ell}) > c\sum_{\ell<2s}\omega(2^{-\ell}) > c\sum_{j<4^s}\tfrac{1}{j}\omega\big(\tfrac{1}{j}\big)\,. \tag{5}$$

We now put

$$f(x)=\sum_\nu 4^{-N_\nu} f_{s_\nu}(2^{N_\nu}x)\,. \tag{6}$$

where the numbers N_ν and s_ν are chosen as follows: if the partial sum $g_{\nu-1}$ of the series (6) is already defined we take

$$N_\nu > N_{\nu-1}+2s_{\nu-1}\,, \tag{7}$$

and then take s_ν so large that

$$\sum_{j<4^{s_\nu}} \tfrac{1}{j}\omega\left(\tfrac{1}{j}\right) > 4^{2N_\nu}\,. \tag{8}$$

Clearly $\omega_f(\delta) < C\sum 4^{-N_\nu}\omega_{f_{s_\nu}}(2^{N_\nu}\delta) < C\omega(\delta)$. Now fix I to be of the form $[p2^{-N_\nu},(p+1)2^{-N_\nu}]$. (7) implies that $\lambda(g_{\nu-1},I)<1$. Denoting by $\{x_j\}$ the extremal points of f_{ν_s} on I in decreasing order and keeping in mind that all further members of (6) vanish at each x_j, we get from (5),(6),(8) that $\lambda(f,I)>4^{N_\nu}$. □

REMARK. As in the previous sections, there is also a version of this theorem pertaining to the order of growth of $\|S_n(f\circ\varphi)\|$. It is possible to prove that for any $s(n)=o(\log n)$ one can construct a function $f\in C(\mathbb{T})$ such that

$$\limsup \frac{1}{s(n)}\big\|S_n(f\circ\varphi)\big\| = \infty$$

for a second category set of homeomorphisms φ. Further, in the category setting one can strengthen this result to get

$$\limsup \frac{1}{s(n)}\big|S_n(f\circ\varphi)(t)\big| = \infty$$

on a set S of (t,φ)'s which is of second category in $\mathbb{T}\times C(\mathbb{T})$ and $S\cap\{t=t_0\}$ is of second category in $C(\mathbb{T})$ for every t_0.

This kind of extension cannot happen in measure setting, due to the Carleson convergence theorem.

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