Lemma 1. *At time n*

$$\sum_{x\in\mathbb{Z}^d} \frac{T_n(x)}{|x|} \simeq \sum_{x\in\mathbb{Z}^d} \mathbf{1}\{T_n(x)\neq 0\}$$

where $T_n(x)$ is the local time of excited-to-the-center at time n and place x, and where $\simeq$ is your favourite approximate equality.

Proof. Use the obvious martingale, and the difference would be $\approx \sqrt{n}$. There's some mucking to do at 0, but it's not serious. □

Lemma 2. *Fix a time n and a radius r, and let $A \subset \partial B(r)$ be the set of sites visited by the process at time n ($\equiv \{x \in \partial B(r) : T_n(x) > 0\}$). Then*

$$\mathbb{P}\left(|A| < r^{d-1-\epsilon} \text{ and } \sum_{x\in A} T_n(x) > \frac{r}{10}|A|\right) < e^{-cr}.$$

This is the main lemma and its proof is quite complicated. You have to count over A and show that for a given A this probability is $< e^{-cr|A|}$. Basically there is a "volume exhaustion" argument where you count excursions from A to $\partial B(r)\setminus A$ and show that the number of cookies "needed" to divert them all is larger than the volume available. For example, if A is a singleton this is trivial: in r walks you eat the few neighbouring cookies very quickly and then have exponentially small probability to never hit any other point of $\partial B(r)$ in the remaining $r - C$ visits.

Let us demonstrate the argument for A a spherical cap (kippa).

Lemma 3. *With the same n, r and A, and with some point $a \in \partial B(r)$ and some parameter $s \ll r$*

$$\mathbb{P}\left(A = B_a(s)\cap \partial B(r) \text{ and } \sum_{x\in A} T_n(a) > rs^{d-1}\right) < e^{-cr}.$$

(I hope the argument works for $s = r^{1-\epsilon}$ but the proof below might only work for much smaller s)

Proof. We claim that for any cookie configuration and any $v \in A$,

$$\mathbb{P}^v(\text{Either } E \text{ hits } \partial B(r)\setminus A \text{ or eats } cs \text{ cookies before time } cs^2) \geq c. \tag{1}$$

The argument is a coupling with random walk: random walk has positive probability to, when starting from v, go distance $\approx s$ inside, and then $\approx s$ outside, piercing $\partial B(r)$ at a distance $\approx s$ from A. If it does that, it has to be shifted at least s places to either avoid $\partial B(r)$ or hit is at a. This shows (1). We can improve this to check what happens before the next hitting of A by paying another s:

$$\mathbb{P}^v(\text{Either } E \text{ hits } \partial B(r)\setminus A \text{ or eats } cs \text{ cookies before the next visit to } A) \geq c/s.$$

This is because for the first step (going inside $\approx s$ steps) we can add the requirement that $\partial B(r)$ is not visited by the simple random walk and pay $1/s$. But because the two processes can be coupled so that the excited is always more inner than the random, the excited has probability $> c/s$ to do the first part without returning to $\partial B(r)$.

Now, we assumed $T_n(a) > rs^{d-1}$ so we have rs^{d-1} attempts, so this event happens $\geq crs^{d-2}$ times with high probability. But there are only s^d cookies in the relevant part, so only s^{d-1} excursions might eat s cookies, so as long as $r \gg s$, one must have also excursions that exit A (except for the negligible event). □

Theorem. *Excited to the center is recurrent.*

Proof. By lemma 1 one must have layers where the average number of visits is $\simeq r \cdot$the number of visited vertices. By lemma 2, such layers must be fully visited. This means that one has at least $r^{d-\epsilon}$ excursions from the layer inside. Coupling the process with an appropriate Bessel process shows that each excursion has probability $> r^{1-d}$ to hit 0. Hence the process return to 0 at least $r^{1-\epsilon}$ times. □