COVER TIME FOR BRANCHING RANDOM WALK ON GRAPHS

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ABSTRACT. The cover time of d-branching random walk on the first n levels of a d-regular tree is a.s. $n - O(\log n)$.

1. INTRODUCTION

The cover time of simple random walk on finite graphs is well studied and has many algorithmic applications [1]. Consider *branching random walk*(BRW), where each particle branches to d independent particles each jumps to a uniformly chosen neighbor. The speed of BRW on $\mathbb{Z}$ is well studied subject started with several papers by Biggins, see e.g. [4]. Hopefully (or maybe already) BRW can have some algorithmic uses too.

It is easy to see that if G is a finite graph with a uniform bound on the degree, then a.s. the cover time of d-BRW, $d \geq 2$, is proportional to the diameter with a constant depending only on the degree see e.g. the arguments in [2].

The theorem below implies that if the branching $d \geq 2$ is at least the maximal degree, then with high probability the cover time equals the radius of the graph up to lower order corrections.

Theorem. *Let $d \geq 2$, and examine a d-BRW on a d-regular tree. Then a.s. it covers up to time n a complete subtree of height $n - O(\log n)$.*

Note that for $(d-\epsilon)$-BRW, the cover will take time Cn with $C > 1$, for any $\epsilon > 0$.

Possible generalizations are to consider asymptotic cover time of tree indexed random walk on trees, where the time tree has branching number d, and the space tree admit branching number d'. See [2] for the study of tree index random walks and the book [5] for background. Variants of branching random walk and tree index random walks were used in [3, 6] to study embedding of trees into graphs..

A natural tree to consider is that of critical branching process conditioned upon surviving.

2. PROOF

Proof. Examine the number of walkers at the starting vertex (which we nickname "the root") at even times. Clearly the expected number of walkers multiplies by d at every step (step being $t \to t+2$). Therefore, by the a.s. exponential growth of supercritical branching process conditioned on surviving, with positive probability, after $T := \lfloor C \log n \rfloor$ steps one has n^4 walkers at the root. Examine now the trajectory of these walkers. Assume that at time t there are $k(v)$ walkers at a node v, and let w be a neighbor of v. Then for some $C_1, c_2 > 0$

$$\mathbb{P}(\text{there are } k(v) - \lambda\sqrt{k(v)} \text{ walkers at } w \text{ at time } t+1) \leq C_1 e^{-c_2\lambda^2}.$$

Setting $\lambda = C\sqrt{n}$ for some sufficiently large C gives that this probability is so small that by summing the probabilities this event never happens for any $t \leq n$ with probability going to 1, for any v in the tree. Conditioned on the event that for all v, for a sequence of $k(v)'s$ that be determined below, this implies that the number of walkers at a node in height h at time $T + h$ is bounded below by a_h defined by

$$a_0 = n^4 \quad a_i = a_{i-1} - C\sqrt{na_{i-1}}.$$

A simple induction shows that $a_h \geq \frac{1}{2}h^4$ for all $h \leq n$ and n sufficiently large, we are done. □

Remark. For a $d+1$-branching walk on a d-regular tree, the subtree covered is in fact $n - O(1)$. The proof is similar, but it is enough to initialize the process with Cd^4 walkers in the root (rather than n^4) — the a_i-s would satisfy $a_i = a_{i-1}(1 + 1/d) - C\sqrt{da_{i-1}}$ and it is again easy to see that $a_i \to \infty$.

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