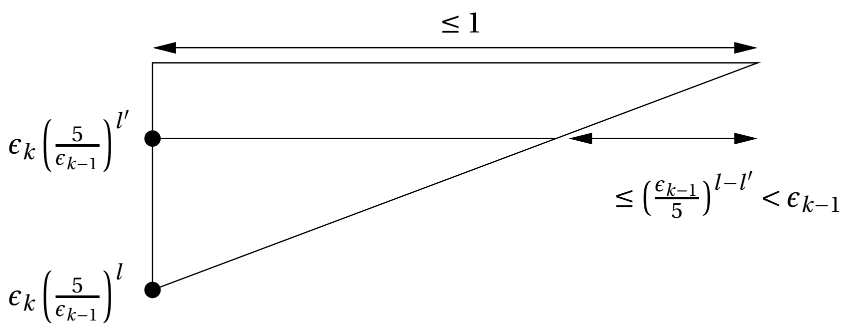

FIGURE 1. Slopes

For a measure μ on $\mathbb{R}$, let K_n be the convex hull of n independent samples of $\mu \otimes \mu$, and let V_n be the number of vertices of K_n.

Claim. There exists a measure μ such that $\mathbb{E}V_n \leq C$ for all n.

Proof sketch. Let $\epsilon_0 = \frac{1}{10}$ and let $\{\epsilon_k\}_{k\geq 1}$ be defined inductively by

$$\epsilon_k = \left(\frac{\epsilon_{k-1}}{10}\right)^{2^k}$$

(i.e. a very fast decreasing sequence — one may take a faster sequence too). The measure will be absolutely continuous with respect to the Lebesgue measure, with the density constant on certain intervals which we will now define. Let $k \geq 0$ and let $l \in \{0, \ldots, 2^k - 1\}$. We define the interval $I_{k,l}$ using

$$I_k = [a_k, a_k + \epsilon_{k+1}] \qquad a_k = 2^{-k-2} - \epsilon_k \left(\frac{5}{\epsilon_{k-1}}\right)^l.$$

Now define μ such that $\mu(I_k) = 2^{-(2^k+l+2)}$. Put an identical mass at $1 - I_k = [1 - a_k, 1 - a_k - \epsilon_{k+1}]$. In words, the measure is defined in two steps: in the first put "strips" of width 2^{-2^k} at place 2^{-k} with μ approximately 2^{-2^k}. Then divide each strip into much thinner lines of width ϵ_{k+1}, located at an increasing exponential sequence.

Now, the claim is that for n, the convex hull K_n has approximately 20 vertices. To locate them let m satisfy $2^{m+3} < n \leq 2^{m+4}$. Write $m = 2^k + l$ with $l < 2^k$ as above. Finally write $l = 2^r + s$ with $s < 2^r$. Now, 8 of the vertices will be in $I_{k,l} \times I_{0,0}$ and its images with respect to reflections through the middles and diagonals of the cube. 8 with be found in $I_{k,0} \times I_{r,s}$ and its friends. And 4 more in $I_{k-1,l/2} \times I_{k-1,l/2}$ (if l is odd take $(l+1)/2$ and $(l-1)/2$ instead). This is not a precise claim, inside each such rectangle one may find more than one vertex, but the expected number is constant. It is straightforward to verify that each such rectangle has $\mu \otimes \mu = 2^{-m}/16$ so one may expect about one point in each. We do need to verify that all other points are in the convex hull of these points.

Let us examine for example points in $I_{k,l'} \times [0,1]$ for some $l' < l$ (for $l' > l$ the measure is $\ll 1/n$ so one does not expect points there, roughly speaking). The diagonal from (any point of) $I_{k,l} \times I_{0,0}$ to any point of $I_{k,0} \times I_{r,s}$ has slope $\leq \epsilon_k (5/\epsilon_{k-1})^l$ and therefore (examine figure 1) any point in $I_{k,l'} \times [0,1]$ which wants any hope to be a vertex, must be at $I_{r,s}$ or higher, because ϵ_{k-1} is the smallest distance between intervals possible until the k^{th} strip. The expected number of such points is

$$= n\mu(I_{k,l'})\mu(I_{r,s}) = n2^{-2^k-l'-2-2^r-s-2} = n2^{-m-4-l'} \leq 2^{-l'}$$

so the total extra vertices beyond this line is bounded.

I did not yet do detailed calculations for the other strip (from $I_{k,o} \times I_{r,s}$ to $I_{k-1,l/2} \times I_{k-1,l/2}$) but I belive they are quite similar. □

Why is the double splitting (to strips and then to lines) needed? Is it needed? I don't know. It seems that by a little tweeking one may make $V_n \to 16$ almost surely as $n \to \infty$. Is it possible to do better than 16?